Computing the dispersion diagram and the forced response of periodic elastic structures using a state-space formulation

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Abstract
In the context of acoustic black hole investigations, recent works have proposed the use of a spatial state-space formulation for one-dimensional elastic waveguides. The boundary value problem is thus transformed into an initial value problem. Given that the state (displacements and forces) cannot be known a priori at any given boundary, but the impedance can, the state-space problem is recast into an impedance formulation, in the form of a Riccati equation. In this paper, this formulation is extended to compute the transfer matrix of a periodic cell of a one-dimensional elastic waveguide. With this transfer matrix, not only can the dispersion diagram be computed, but also the forced response of the finite structure. The simple case of an elastic rod is used to illustrate the proposed method. The dispersion diagram is verified with the plane wave expansion method, and the forced response is verified with the spectral element method. Numerical results show that the proposed method is an efficient way to characterize wave propagation in period elastic structures.

1 Introduction

Linear elastodynamic problems are usually formulated as boundary value problems, which may be solved analytically or numerically [1]. Analytical solutions are often derived in the frequency domain, which can be referred to as spectral solutions. For simple structures, it is straightforward to derive solutions in the form of finite elements in the frequency domain. This semi analytical approach is known as spectral element method (SEM) [2]. In SEM, a global dynamic stiffness matrix can be assembled via the direct stiffness method, commonly used in finite element analysis [1].

Deriving a spectral element for a straight homogeneous rod with constant material and geometrical properties along its length is a straightforward process. However, this task can be awkward for rods with varying properties. Analytical solutions exist for only a few cases, such as rods with linearly varying (tapered) and exponentially varying geometrical properties.

Some authors have proposed a solution method (see [3] and references therein) for acoustic waveguides (Helmholtz spectral equation), in which the problem is reformulated in the state-space framework, where the state consists of the pressure and the particle velocity along the tube. This approach transforms the boundary value problem into an initial value problem. Given that one does not known the initial condition, i.e. the state at one end of the waveguide, the authors in [3] have reformulated the problem in terms of the impedance, the ratio between pressure and velocity.
The problem stated in terms of the impedance yields a Riccati matrix equation that can be solved analytically, for simple problems, or numerically, for more complex problems. The Riccati differential equation (RDE) plays an important role in many engineering science applications [4]. This type of equation [5, 6], which is quadratic and, thus, nonlinear, can be reduced to a linear system of twice its size that can be efficiently solved using numerical algorithms [7, 8].

Georgiev et al. [9] recently used this method to solve the problem of a beam with a varying cross-section. The thickness, which decreases with a power law profile, can be tailored to minimize wave reflection by slowing the propagation speed and eventually stopping the propagation. This creates an anechoic termination, which is called an acoustic black hole (ABH). The authors in [9] have only computed the impedance of the beam. The solution technique consists of writing the elastodynamic equations as a state-space equation in the frequency domain, as a function of the spatial variable only. Restating the problem for an impedance variable, a Riccati equation is formulated and numerically solved.

In this paper, this method is complemented by a back propagation of the force and displacement (state) at one end of the rod, to compute the full state at the other end. From the relations between these states, it is shown how to compute the dynamic stiffness or the transfer matrix of a finite rod with arbitrarily varying geometrical and material properties. With these matrices, the forced responses may be computed.

The proposed method is particularly interesting for periodic waveguides. Given a periodic rod cell, the dispersion relation (also known as dispersion diagram) can be computed using the transfer matrix by applying the Floquet-Bloch periodicity condition [10]. The dispersion relation shows the frequency bands where the wavenumber becomes complex, which indicates a band gap, where there is no propagation and, therefore, no normal modes can exist. Furthermore, given the dynamic stiffness matrix of the rod cell, the global dynamic stiffness of the built up structure can be easily assembled, allowing the computation of the forced response.

With the proposed technique, it is straightforward to analyze rods with varying cross-section, which is useful, for instance, to optimize the waveguide shape aiming at creating band gaps to reduce or enhance vibration energy propagation. The proposed method can be extended to treat other structural one-dimensional waveguides, such as beams.

2 Modeling Methods

This section presents the proposed method. The elementary rod theory is used, but the technique can be extended to other one-dimensional structures. The problem is written in the state-space form, leading to a Riccati differential equation in terms of the mechanical impedance. Starting from a known impedance at one end, the impedance at the other end is obtained. For a given input force, the state can be obtained for the whole periodic rod cell, using the computed impedance. With the obtained state, the transfer matrix and the dynamic stiffness matrix are derived.

The proposed method is numerically verified using the dispersion diagram of a periodic rod cell with varying properties, and also the forced response of a finite periodic rod. The PWE method [11] is used to compute the dispersion diagram, while the SEM [12] is used for the computation of both the dispersion relation and the forced response.

Given a periodic one dimensional structure, the SEM computes a frequency-dependent dynamic stiffness matrix of one element, which can be assembled as a global stiffness matrix. Using the dynamic stiffness matrix of one periodic element, the transfer matrix is obtained, whose eigenvalues yield the dispersion relation. However, the SEM is limited to simple geometries such as the homogeneous rod and the linearly tapered rod, while the PWE method allows the computation of the dispersion relation for more complex geometries. First, the SEM and the PWE method are briefly reviewed. Then, the proposed approach is presented. All the methods in this paper can be applied to symmetric and to nonsymmetric cells as well.
2.1 Spectral Element Method

This section derives the spectral elements for homogeneous and linearly tapered rods. Using these spectral elements, one with a trapezoidal and the other with a rectangular profile, two types of periodic rods are modeled, as illustrated in Figure 1. The trapezoidal profile is generated with two coupled tapered elements, and the rectangular profile with homogeneous elements with different cross-sections.

![Figure 1: Symmetric cell shapes built with rectangular and tapered spectral elements.](image)

Figure 2 shows a two-node tapered rod spectral element with linearly varying cross-sectional area. In the trapezoidal case, the tapered spectral element developed by [13], for acoustic duct element, was used in [14]. The homogeneous case can be found in [2]. The tapered element is described by the following equation

\[ A(x) = \epsilon(x + \xi) \quad \text{with} \quad \xi = \frac{l A_0}{A_l - A_0} \quad \text{and} \quad \epsilon = \frac{A_0}{\xi} \]

(1)

where \( l \) is the length of the element, \( A(x) \) is the varying cross-sectional area, in which \( A_0 \) and \( A_l \) are the areas at the positions \( x = 0 \) (the smaller edge) and \( x = l \) (the larger edge), respectively. The elementary rod theory considers a slender structure that supports only axial stresses, neglecting the lateral contraction (Poisson’s effect). As shown in [10], the equation of motion for a tapered rod is given by

\[ \frac{\partial}{\partial x} \left[ EA(x) \frac{\partial u(x,t)}{\partial x} \right] = \rho A(x) \frac{\partial^2 u(x,t)}{\partial t^2} \]

(2)

with \( u(x,t) \) the axial displacement, \( \rho \) the mass density, and \( E = \bar{E}(1 + j\eta) \) the complex (or dynamic) Young’s modulus, to account for energy dissipation, where \( \bar{E} \) is the Young’s modulus, \( \eta \) is the loss factor, and \( j = \sqrt{-1} \).
Using the expression for $A(x)$ given by (1), the elastodynamic equation (2) becomes

$$
(x + \xi) \frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial u(x, t)}{\partial x} - \frac{p}{E} (x + \xi) \frac{\partial^2 u(x, t)}{\partial t^2} = 0
$$

The general form of the solution $u(x, t)$, for the elastodynamic equation (3), is assumed to be expressed as

$$
u(x, t) = \hat{u}(x)e^{j\omega t}
$$

where $\hat{u}(x) := \hat{u}(x, \omega)$ is the displacement in the frequency domain. Thus, substituting (4) in (3), one obtains, after some manipulation

$$
d^2 \hat{u}(x, \omega) + \frac{1}{(x + \xi)} \frac{d\hat{u}(x, \omega)}{dx} + k^2 \hat{u}(x, \omega) = 0
$$

where $k = \omega\sqrt{\rho/E}$ is the wavenumber. As shown in [15], a Bessel type solution for (5) is given by

$$
\hat{u}(x) = \alpha_1 J_0(\gamma) + \alpha_2 Y_0(\gamma), \quad \gamma = k(x + \xi)
$$

where $\gamma$, $\alpha_1$, and $\alpha_2$ are constants determined from the boundary conditions, $J_0$ and $Y_0$ are, respectively, Bessel functions of first and second kind, both of order zero. The displacement boundary conditions for the two-node element, shown in Figure 2 are

$$
\hat{u}_1 = \hat{u}(0) = \alpha_1 J_0(k\xi) + \alpha_2 Y_0(k\xi)
$$

$$
\hat{u}_2 = \hat{u}(l) = \alpha_1 J_0(kl + k\xi) + \alpha_2 Y_0(kl + k\xi)
$$

where, $\hat{u}_1$ and $\hat{u}_2$ are the nodal displacements. This equation can be rewritten in matrix form as

$$
\hat{u} = \Phi \alpha, \quad \text{with} \quad \hat{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}, \quad \Phi = \begin{bmatrix} J_0(k\xi) & Y_0(k\xi) \\ J_0(kl + k\xi) & Y_0(kl + k\xi) \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}
$$

Noting that the rod axial force is given by

$$
\dot{q}(x) = EA(x) \frac{\partial \hat{u}(x)}{\partial x}
$$

and evaluating $\dot{q}(x)$ at the element boundaries, the following equations are obtained

$$
\dot{q}_1 = \dot{q}(0) = EA_0 \frac{\partial \hat{u}(x)}{\partial x} \bigg|_{x=0} = -k EA_0 [\alpha_1 J_1(k\xi) + \alpha_2 Y_1(k\xi)]
$$

$$
\dot{q}_2 = \dot{q}(l) = EA_1 \frac{\partial \hat{u}(x)}{\partial x} \bigg|_{x=l} = -k EA_1 [\alpha_1 J_1(kl + k\xi) + \alpha_2 Y_1(kl + \xi)]
$$

where $J_1$ and $Y_1$ are, respectively, Bessel functions of first and second kind, both of first order. The above equation in matrix form is given by

$$
\dot{q} = \Psi \alpha, \quad \text{with} \quad \Psi = -k E \begin{bmatrix} A_0 J_1(k\xi) & A_0 Y_1(k\xi) \\ A_1 J_1(kl + k\xi) & A_1 Y_1(kl + k\xi) \end{bmatrix}
$$

Now, solving (6) for $\alpha$ and substituting the result in (8), one obtains

$$
\dot{q} = \hat{K}(\omega) \hat{u}, \quad \hat{K}(\omega) = \Psi \Phi^{-1}
$$

where $\hat{K}(\omega)$ is a dynamic stiffness matrix of the two-node element with linearly varying cross-sectional area.

The trapezoidal element can now be assembled by interchanging the terms of the principal diagonal of the dynamic stiffness matrix of one element and coupling it with a regular element, condensing the internal node.

As shown in [10], the transfer matrix can be evaluated from the dynamic stiffness matrix as follows

$$
\hat{T}(\omega) = \begin{bmatrix} -K_{12}^{-1} K_{11} & -K_{12}^{-1} \\ K_{21} & K_{22} \end{bmatrix}
$$

where $K_{ij}$, for $i, j = 1, 2$, are the entries of the $2 \times 2$ stiffness matrix $\hat{K}(\omega)$. 
### 2.2 Plane Wave Expansion Method

The sinusoidally varying element is modeled by the plane wave expansion (PWE) method. Figure 3 illustrates the geometry of a symmetric cell shape with sinusoidal cross-sectional area. The rod width is constant in the direction perpendicular to the figure, so that the cross-sectional area varies sinusoidally.

![Figure 3: Symmetric cell shape with sinusoidal cross-sectional area.](image)

Applying the Fourier transform on (2), which is the governing equation for longitudinal vibrations of a straight rod, one obtains

\[
\frac{\partial}{\partial x} \left[ EA(x) \frac{\partial \hat{u}(x, \omega)}{\partial x} \right] + \omega^2 \rho A(x) \hat{u}(x, \omega) = 0 \tag{10}
\]

It is assumed that the rod has a periodic cross-sectional area variation given by \( A(x) = A(x + a) \), where \( a \) is the lattice parameter, i.e., the unit cell length.

Applying the Floquet-Bloch periodicity condition of the solution in \( x \), and considering one-dimensional wave propagation, one obtains

\[ \hat{u}(x) = \tilde{u}(x) e^{j k x} \]

with the Bloch wave amplitude \( \tilde{u}(x) \) periodic of period \( a \), i.e., \( \tilde{u}(x + a) = \tilde{u}(x) \), \( k \) is the Bloch wave vector (here being scalar), also known as wavenumber. The wave vector has its value within the first irreducible Brillouin zone (FIBZ), in the reciprocal space, \([0, \pi/a]\), or within the first Brillouin zone (FBZ), \([-\pi/a, \pi/a]\).

Expanding the Bloch wave amplitude \( \tilde{u}(x) \) as a Fourier series in the reciprocal space, yields

\[
\hat{u}(x) = \left( \sum_{m=-\infty}^{+\infty} u_m e^{j g_m x} \right) e^{j k x} = \sum_{m=-\infty}^{+\infty} u_m e^{j (k + g_m) x} \tag{11}
\]

where \( u_m \) are the coefficients of the Fourier series of \( \tilde{u}(x) \) and \( g_m = 2\pi m/a \) is the reciprocal lattice vector. Note that \( g_m \) is a constant, since a one-dimensional periodicity is considered. Furthermore, the cross-sectional area can also be expanded as Fourier series in the reciprocal space as

\[ A(x) = \sum_{n=-\infty}^{+\infty} A_n e^{j g_n x} \tag{12} \]

where \( g_n = 2\pi n/a \). Note that Fourier series coefficients \( A_n \) in (12) can be computed using

\[ A_n = \frac{1}{a} \int_{-a/2}^{a/2} A(x) e^{-j g_n x} \, dx \]

Substituting (11) and (12) in (10), gives

\[
\sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left( EA_n (k + g_m)(k + g_m + g_n) - \omega^2 \rho A_n \right) u_m e^{j (k + g_m + g_n) x} = 0 \tag{13}
\]
Multiplying (13) by \( e^{-j(k+gr)x} \), with \( gr = 2\pi r/a \), and integrating from \(-a/2\) to \(a/2\), yields

\[
\sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left( EA_n(k + g_m)(k + g_m + g_n) - \omega^2 \rho A_n \right) u_m \frac{1}{a} \int_{-a/2}^{a/2} e^{j(g m + g n - gr)x} \, dx = 0 \tag{14}
\]

Given that

\[
\frac{1}{a} \int_{-a/2}^{a/2} e^{j(g m + g n - gr)x} \, dx = \begin{cases} 
1 & \text{if } n = r - m \\
0 & \text{otherwise}
\end{cases}
\]

one can rewrite (14) as

\[
\sum_{m=-\infty}^{+\infty} \left( EA_{r-m}(k + g_m)(k + gr) - \omega^2 \rho A_{r-m} \right) u_m = 0
\]

Equivalently

\[
\sum_{m=-\infty}^{+\infty} EA_{r-m}(k + g_m)(k + gr) u_m = \lambda \sum_{r=-\infty}^{+\infty} \rho A_{r-m} u_m, \quad \lambda = \omega^2 \tag{15}
\]

which is a system with an infinite amount of equations. Thus, to solve this system, one can truncate the Fourier series to the first \( M \) terms, i.e., \( r, m \in [-M, \ldots, M] \in \mathbb{Z} \), such that (15) can be rewritten as

\[
Bu = \lambda Cu \tag{16}
\]

where the coefficients of vector \( u \) are \( u_m \) and the coefficients of matrices \( B \) and \( C \) are given by

\[
B_{rm} = EA_{r-m}(k + g_m)(k + gr), \quad C_{rm} = \rho A_{r-m}
\]

Notice that (16) represents a generalized eigenvalue problem on \( \lambda \) and should be solved for each \( k \), within FBZ or FIBZ.

### 2.3 State-Space Formulation

This section presents the proposed method. It shows how to write the elastodynamic equations for a rod in a state-space formulation, to rewrite the problem in terms of the mechanical impedance and to solve a Riccati equation to obtain the impedance at one end, given a zero impedance at the other end (free end). It also illustrates how to obtain the transfer matrix and the dynamic stiffness matrix from the impedance. The results obtained with this method will be referred to as state-space formulation (SSF) in the numerical section.

Using (10) and (7), one can write the set of state-space equations

\[
\frac{\partial \hat{q}(x)}{\partial x} = -\omega^2 \rho A(x) \hat{u}(x) \quad \text{and} \quad \frac{\partial \hat{u}(x)}{\partial x} = \frac{\hat{q}(x)}{EA(x)}
\]

which can be written in matrix form as

\[
\frac{\partial \hat{p}}{\partial x} = H \hat{p} \tag{17}
\]

with the state \( \hat{p}(x) := \hat{p}(x, \omega) \) and \( H(x, \omega) \) having the following expressions

\[
\hat{p}(x) = \begin{bmatrix} \hat{u}(x) \\ \hat{q}(x) \end{bmatrix} \quad \text{and} \quad H(x, \omega) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{EA(x)} \\ -\omega^2 \rho A(x) & 0 \end{bmatrix}
\]
Note that this is a system of linear ordinary differential equations of the first order, where the system parameters may vary along \( x \). It is straightforward to solve this system numerically for any type of parameter variation along \( x \), if the state initial condition is known. However, the initial condition is not known, since the boundary condition at any edge of the structure is given by either the force (Neumann condition), the displacement (Dirichlet condition), or the mechanical impedance (mixed condition). Thus, to overcome this issue, one can rewrite the problem (see [9]) in terms of the mechanical impedance \( \hat{z}(x) \), which is the relation between the state variables \( \hat{u} \) and \( \hat{q} \) given by

\[
\hat{q}(x) = j\omega \hat{z}(x) \hat{u}(x)
\]

Substituting (18) into (17), one obtains the following set of equations

\[
\frac{\partial \hat{u}(x)}{\partial x} = H_{11} \hat{u}(x) + H_{12} \hat{q}(x)
\]

and

\[
\frac{\partial \hat{z}(x) \hat{u}(x)}{\partial x} = H_{21} \hat{u}(x) + H_{22} \hat{q}(x)
\]

which leads to

\[
j\omega \left( \frac{\partial \hat{z}(x)}{\partial x} \hat{u}(x) + \hat{z}(x) H_{11} \hat{u}(x) + \hat{z}(x) H_{12} j\omega \hat{z}(x) \hat{u}(x) \right) = H_{21} \hat{u}(x) + H_{22} j\omega \hat{z}(x) \hat{u}(x)
\]

Since \( \hat{u}(x) \) cannot be identically zero, one finally obtains the following Riccati equation

\[
\frac{\partial \hat{z}(x)}{\partial x} + \hat{z}(x) H_{11} - H_{22} \hat{z}(x) + j\omega \hat{z}(x) H_{12} \hat{z}(x) = (j\omega)^{-1} H_{21}
\]

To solve the above Riccati differential equation, it is necessary an initial condition, which can be taken to be \( \hat{z}(0) = 0 \). This condition on a free end of the structure corresponds to the force \( \hat{q}_0 = 0 \), and a displacement \( \hat{u}_0 \) unknown but different from zero. Once the impedance \( \hat{z}(L) \), the solution of the Riccati equation at the terminal position \( x = L \), is computed using the initial condition \( \hat{z}(0) = 0 \), the displacement \( \hat{u}(L) \) can be computed by considering that the external force in (18) is given by \( \hat{q}(L) = 1 \).

With \( \hat{q}(L) \) and \( \hat{u}(L) \) computed, the state \( \hat{p}(L) \) can be used as the initial condition in (17) so that the state \( \hat{p}(0) \) can be obtained by a backward integration of the state equations (17). Then, using the states at both ends, given by \( \hat{p}(0) \) and \( \hat{p}(L) \), it is possible to compute the transfer matrix, which relates the states at the extremes of a finite rod, as follows

\[
\hat{p}(L) = \hat{T}(\omega) \hat{p}(0)
\]

where \( \hat{T}(\omega) \) is the transfer matrix of size \( 2 \times 2 \). Using the boundary conditions \( \hat{q}(L) = 1 \), \( \hat{q}(0) = 0 \), \( \hat{u}(L) = \hat{u}_L \), and \( \hat{u}(0) = \hat{u}_0 \), ones obtains

\[
\begin{bmatrix}
\hat{u}_L \\
0
\end{bmatrix} =
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{u}_0 \\
0
\end{bmatrix}
\]

Solving this algebraic system of equations leads to

\[
T_{11} = \hat{u}_L / \hat{u}_0, \quad T_{21} = \hat{u}_0^{-1}
\]

\[
T_{22} = T_{11}, \quad \text{(due to the rod cell symmetry)}
\]

The relation between the off diagonal terms of matrix \( \hat{T} \) can be evaluated using the relation between the transfer matrix and the symmetric (due to reciprocity) dynamic stiffness matrix, given by (9), and imposing \( T_{11} = T_{22} \) for a symmetric cell. It can be shown that

\[
T_{12} = (T_{11}^2 - I) T_{21}^{-1}
\]

Once the transfer matrix \( \hat{T}(\omega) \) is computed, the dynamic stiffness matrix is readily obtained using (9).
If the structure cell is not symmetric, $T_{11} \neq T_{22}$, the above formulation is unable to compute $T_{12}$. In order to overcome this issue, the initial condition is applied at the other end, $x = L$ and the impedance is integrated in the opposite direction up to $x = 0$. Repeating the procedure done earlier, one obtains $\hat{\rho}(0)$ and $\hat{q}(L)$. Applying the new boundary conditions, given by $\hat{q}(L) = 0$ and $\hat{q}(0) = 1$, the transfer matrix relation becomes

$$
\begin{bmatrix}
\hat{u}^*_L \\
0
\end{bmatrix} =
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{u}^*_0 \\
1
\end{bmatrix}
$$

Writing $T_{12}$ and $T_{22}$ in terms of $T_{11}$ and $T_{21}$, gives

$$
T_{12} = \hat{u}^*_L - T_{11} \hat{u}^*_0, \quad T_{22} = -T_{21} \hat{u}^*_0
$$

where $\hat{u}^*_L$ and $\hat{u}^*_0$ are the new displacement terms obtained. Now, with $T_{11}$ and $T_{21}$ previously obtained, the transfer matrix and, consequently, the dynamic stiffness matrix, are obtained.

3 Numerical Results

In this section, the proposed method is verified for three different structural models by comparison with conventional methods. In these examples, only the variation of the cross-sectional area is considered. The material is homogeneous steel with structural loss factor $\eta = 0.01$, incorporated as a complex Young’s modulus $E = \bar{E}(1 + j\eta)$. The material and geometrical properties used for all the examples are: $L = 0.5$ [m], $E = 210$ [GPa], and $\rho = 7800$ [kg/m$^3$]. Besides the trapezoidal and rectangular cases shown in Figure 1, a case with sinusoidally varying cross-sectional area, shown in Figure 3, is treated. In the latter case, the Fourier series has only 3 nonzero plane waves.

![Figure 4: Three different variations of the cross-sectional area.](image)

The three examples are shown in Figure 4. In the sinusoidal profile: $A_1 = 1 \times 10^{-2}$ [m$^2$] and $A_2 = 2 \times 10^{-2}$ [m$^2$]. In the rectangular profile: $A_1 = 1 \times 10^{-2}$ [m$^2$] and $A_2 = 2 \times 10^{-2}$ [m$^2$]. In the trapezoidal profile: $A_1 = 1 \times 10^{-4}$ [m$^2$] and $A_2 = 3 \times 10^{-2}$ [m$^2$].
The first verification consists in comparing the impedance obtained by solving the Riccati equation using an algorithm based on the Runge-Kutta method with the analytical solution obtained using SEM for rectangular case. The convergence for decreasing integration step sizes is verified by the mechanical impedance, with relative tolerances: $10^{-2}$, $10^{-4}$, and $10^{-9}$ in Figure 5(a), Figure 5(b), and Figure 5(c), respectively.

The convergence is also verified in Figure 6, which presents the absolute value of element $K_{11}$ of the dynamic stiffness matrix, obtained using the SSF method with different relative tolerances, compared to the analytical solution (in black) obtained with SEM.

The dispersion curve for one periodic cell is verified by comparison with both the PWE method and SEM for the rectangular and trapezoidal profiles. For the sinusoidal case, only the PWE is used for comparison, since
there is no spectral element available in the literature for this cross-sectional area variation. The agreement is very good in all cases, see Figure 7, provided the integration step is chosen appropriately.

![Dispersion diagram for the rectangular (a,b) and trapezoidal (c,d) profiles.](image)

The presence of band gaps can be observed in both models in Figure 7. It is noticeable that the band gap is larger for the tapered profile, suggesting that the smoother variation of area can increase the band gap size.

Figure 8 shows the comparison between PWE and the proposed SSF method for the sinusoidally varying cross-section.

![Real part of dispersion diagram evaluated by SSF and PWE for the sinusoidal rod.](image)

The forced response of a finite rod with ten cells is analyzed in Figure 9 considering the free-free boundary condition and a harmonic force $F = 1 \text{ [N]}$ as excitation at the one end, at $x = L$. The response is measured at the other end, at $x = 0$, avoiding the antiresonance and making it easier to observe the band-gap effect.

![Real part of dispersion diagram evaluated by SSF and PWE for the sinusoidal rod.](image)
Figure 9: Rod displacement at $x = 0$ evaluated using SEM and SSF method.

4 Conclusion

A new solution method for computing the band diagram and forced response of one-dimensional structures using a state-space formulation and the solution of a Riccati equation with impedance as variable was proposed and verified by comparison with existing methods. The band diagrams obtained with the proposed method showed good agreement with results obtained by the traditional PWE method. The forced response obtained with the proposed method showed good agreement with results computed using a SEM formulation. A previous works [9] showed the formulation of the one-dimensional problem of a straight beam with varying cross-section as a state-space problem reformulated into a Riccati equation. The present work extended this approach to compute the transfer matrix and the dynamic stiffness matrix, with which both the dispersion diagram (with real and imaginary parts) and the forced response of a periodic finite structure can be computed. Results were shown for a rod, but can be extended for a beam.

Examples of elastic phononic crystal rods with different geometries were computed to illustrate the proposed method and verify it by comparison with the conventional methods SEM and PWE. All the numerical results obtained with the proposed method were shown to be in agreement with the conventional methods.

The proposed method is an efficient way to characterize the wave propagation in periodic one-dimensional elastic structures, and can be used in shape optimization aiming at designing band gaps for passive vibration control.
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