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# Deduction of plastic work rate per unit volume for unified yield criterion and its application

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**Abstract:** A unified linear expression of plastic work rate per unit volume is deduced from the unified linear yield criterion and the associated flow rule. The expression is suitable for various linear yield loci in the error triangle between Tresca's and twin shear stress yield loci on the $\pi$-plane. It exhibits generalization in which the different value of criterion parameter $b$ corresponds to a specific linear formula of plastic work rate per unit volume. Finally, with the unified linear expression of plastic work rate and upper-bound parallel velocity field the strip forging without bulge is successfully analyzed and an analytical result is also obtained. The comparison with traditional solutions shows that when $b=1/(1+\sqrt{3})$ the result is the same as the upper bound result by Mises' yield criterion, and it also is identical to that by slab method with $m=1$, $\sigma_0=0$.

**Key words:** $\pi$-plane; flow rule; unified linear plastic work rate; deduction; application

## 1 Introduction

Nowadays, the studies on metal forming are most of numerical analysis, such as FEM[1−4] and UBEM[5]. However, almost no theoretical analytical solution can be obtained without simplifying the Mises' yield criterion because of its non-linearity. In recent years, one concern regarding linearization of yield criteria has been put forward and become much more extensive. In 1983, YU[6] proposed a linear twin shear stress yield criterion, called TSS criterion for short. HUANG and ZENG[7] deduced its plastic work rate per unit volume in 1989. The works of YU gave us much interesting to apply his criterion in metal forming. However, it showed a greater calculated result than that by Mises' yield criterion[8−9].

By notice of non-linearity of Mises' criterion, YU [10] proposed a unified linear yield criterion, called UY criterion, in which yield criterion parameter $b$ represents the effect of the intermediate principal shear stress on the yield of materials and $0\leqslant b\leqslant 1$. The UY criterion is not a single yield criterion but a series of continuously variable linear yield criteria. Therefore, its application will play significant role in metal forming, and the plastic work rate per unit volume must be the key procedure.

This work is to deduce the linear plastic work rate per unit volume by UY criterion firstly and then apply it in strip forging, and compare its calculated results with traditional solutions.

## 2 Derivation of plastic work rate

The UY criterion is usually presented by

$$\begin{cases} \sigma_1-\dfrac{b\sigma_2}{1+b}-\dfrac{\sigma_3}{1+b}=\sigma_s & \text{if } \sigma_2\leqslant\dfrac{\sigma_1+\sigma_3}{2} \\ \dfrac{\sigma_1}{1+b}+\dfrac{b}{1+b}\sigma_2-\sigma_3=\sigma_s & \text{if } \sigma_2\geqslant\dfrac{\sigma_1+\sigma_3}{2} \end{cases} \tag{1}$$

where $b$ is the yield criterion parameter, $\sigma_1$, $\sigma_2$ and $\sigma_3$ are principal stresses.

Assuming the stress tensor satisfies $f(\sigma_{ij})=0$, then Levy-Mises' flow rule[11] gives

$$\dot{\varepsilon}_{ij}=d\lambda\frac{\partial f}{\partial\sigma_{ij}} \tag{2}$$

From Eq.(2) and the first formula in Eq.(1), there is

$$\dot{\varepsilon}_1:\dot{\varepsilon}_2:\dot{\varepsilon}_3=1:-\frac{b}{1+b}:-\frac{1}{1+b}=\lambda:-\frac{b\lambda}{1+b}:-\frac{\lambda}{1+b}$$

From Eq.(2) and the second formula in Eq.(1), there

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is

$$\dot{\varepsilon}_1:\dot{\varepsilon}_2:\dot{\varepsilon}_3=\frac{1}{1+b}:\frac{b}{1+b}:-1=\frac{\mu}{1+b}:\frac{b\mu}{1+b}:-\mu$$

Because $\lambda\geqslant 0$, $\mu\geqslant 0$, taking linear combination of above two formulas yields

$$\dot{\varepsilon}_1:\dot{\varepsilon}_2:\dot{\varepsilon}_3=(\frac{\mu}{1+b}+\lambda):\frac{1}{1+b}(\mu-\lambda):-(\frac{\lambda}{1+b}+\mu)$$

Since magnitudes of $\lambda$ and $\mu$ are arbitrary, taking $\dot{\varepsilon}_1=\mu/(1+\mathrm{b})+\lambda$, it yields

$$\dot{\varepsilon}_2=\frac{b}{1+b}(\mu-\lambda),\ \dot{\varepsilon}_3=-(\frac{\lambda}{1+b}+\mu) \tag{3}$$

Among them, $\dot{\varepsilon}_{\max}=\dot{\varepsilon}_1$, $\dot{\varepsilon}_{\min}=\dot{\varepsilon}_3$, thus

$$\dot{\varepsilon}_{\max}-\dot{\varepsilon}_{\min}=\dot{\varepsilon}_1-\dot{\varepsilon}_3=\frac{\mu}{1+b}+\lambda+\frac{\lambda}{1+b}\mu=(\lambda+\mu)\left(\frac{2+b}{1+b}\right)$$

so,

$$\mu+\lambda=\frac{1+b}{2+b}(\dot{\varepsilon}-\dot{\varepsilon}_3)=\frac{1+b}{2+b}(\dot{\varepsilon}_{\max}-\dot{\varepsilon}_{\min}) \tag{4}$$

Taking note of Eq.(3) and $\sigma_2=(\sigma_1+\sigma_3)/2$ at the point $E$, as shown in Fig.1, plastic work rate per unit volume is

$$D(\dot{\varepsilon}_{ij})=\sigma_1\dot{\varepsilon}_1+\sigma_2\dot{\varepsilon}_2+\sigma_3\dot{\varepsilon}_3=\sigma_1\left(\frac{\mu}{1+b}+\lambda\right)+\frac{\sigma_1+\sigma_3}{2}\left[\frac{b}{1+b}(\mu-\lambda)\right]+\sigma_3\left[-(\frac{\lambda}{1+b}+\mu)\right]$$

so,

$$D(\dot{\varepsilon}_{ij})=(\sigma_1-\sigma_3)(\mu+\lambda)\frac{1+b/2}{1+\mathrm{b}} \tag{5}$$

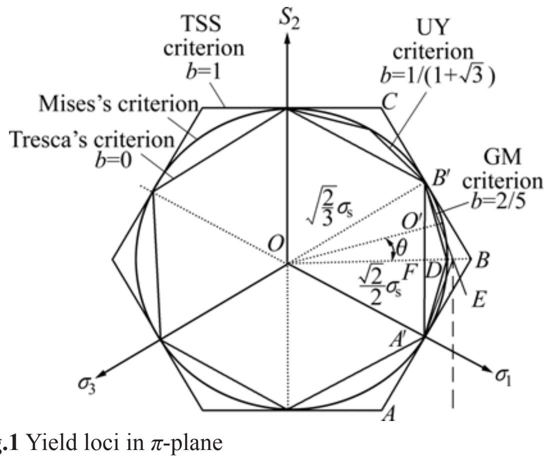


**Fig.1** Yield loci in $\pi$-plane

Simultaneously solving formulas in Eq.(1) as angular point $E$ yields

$$\sigma_1\left(1+\frac{1}{1+b}\right)-\sigma_3\left(\frac{1}{1+b}+1\right)=2\sigma_{\mathrm{s}}$$

Then,

$$\sigma_1-\sigma_3=\frac{1+b}{1+b/2}\sigma_{\mathrm{s}},\quad 0\leqslant b\leqslant 1 \tag{6}$$

Substituting Eqs.(6) and (4) into Eq.(5), the plastic work rate done per unit volume for UY criterion is derived by

$$D(\dot{\varepsilon}_{ij})=\frac{1+b}{2+b}\sigma_{\mathrm{s}}(\dot{\varepsilon}_{\max}-\dot{\varepsilon}_{\min}) \tag{7}$$

## 3 Generalization of Eq.(7)

Substituting $b$=1 into Eq.(7) yields the specific plastic work rate per unit volume for TSS criterion as follows:

$$D(\dot{\varepsilon}_{ij})=\frac{2}{3}\sigma_{\mathrm{s}}(\dot{\varepsilon}_{\max}-\dot{\varepsilon}_{\min}) \tag{8}$$

Substituting $b$=0 into Eq.(7), the specific plastic work rate per unit volume for Tresca's criterion is

$$D(\dot{\varepsilon}_{ij})=\frac{\sigma_{\mathrm{s}}}{2}(\dot{\varepsilon}_{\max}-\dot{\varepsilon}_{\min}) \tag{9}$$

Substituting $b$=2/5=0.4 into Eq.(7), the specific plastic work rate for geometric midline(GM) criterion [12] also results in

$$D(\dot{\varepsilon}_{ij})=\frac{7\sigma_{\mathrm{s}}}{12}(\dot{\varepsilon}_{\max}-\dot{\varepsilon}_{\min}) \tag{10}$$

In the same way, substituting $b$=1/3 and $b$=0.529 into Eq.(7) yields the single plastic work rate for mean yield (MY) criterion[13] and equi-area(EA) criterion[11]. They are respectively

$$D(\dot{\varepsilon}_{ij})=\frac{4}{7}\sigma_s(\dot{\varepsilon}_{\max}-\dot{\varepsilon}_{\min}) \tag{11}$$

$$D(\dot{\varepsilon}_{ij})=\frac{\sqrt{3}\pi}{9}\sigma_s(\dot{\varepsilon}_{\max}-\dot{\varepsilon}_{\min}) \tag{12}$$

Above deduction shows generalization of Eq.(7). It is not a single plastic work rate per unit volume, but a series of linear plastic work rate expressions corresponding to different specific yield criteria with $b$ values from 0 to 1. So, it can be called unified or generalized linear plastic work rate per unit volume.

## 4 Representation on $\pi$-plane

The projection of the principal stress components on the $\pi$-plane[14] is shown in Fig.2, where

$$\begin{cases} a=\sqrt{\frac{2}{3}}\sigma_1\cos 30^\circ-\sqrt{\frac{2}{3}}\sigma_3\cos 30^\circ=\frac{\sigma_1-\sigma_3}{\sqrt{2}} \\ b=\sqrt{\frac{2}{3}}\sigma_2-\sqrt{\frac{2}{3}}(\sigma_1+\sigma_3)\sin 30^\circ=\frac{2\sigma_2-\sigma_1-\sigma_3}{\sqrt{6}} \\ r^2=a^2+b^2=2J_2'=\sigma_1'^2+\sigma_2'^2+\sigma_3'^3 \\ \theta=\tan^{-1}\frac{b}{a}=\tan^{-1}\frac{2\sigma_2-\sigma_1-\sigma_3}{\sqrt{3}(\sigma_1-\sigma_3)}=\tan^{-1}\frac{\mu}{\sqrt{3}} \end{cases} \tag{13}$$

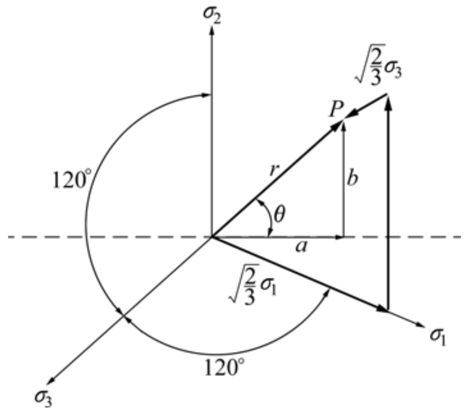


**Fig.2** Polar coordinates of point $P$ on $\pi$-plane

and $\dfrac{2\sigma_2-\sigma_1-\sigma_3}{\sigma_1-\sigma_3}$ is lode stress parameter[14].

As shown in Fig.1, $\theta$ is also the included angle between any linear yield locus (hypotenuse $B'E$) with Tresca's locus (right angled side $B'F$) in error triangle $FB'B$ on $\pi$-plane. As shown in Fig.3, sides of the angle are perpendicular to those of $\theta$=π/6, $B'F$=$\dfrac{\sqrt{2/3}}{2}\sigma_s$, for any $\theta$ in the triangle $FB'B$:

$$\tan\theta=\frac{FE}{B'F}=\frac{\sqrt{6}FE}{\sigma_s} \tag{14}$$

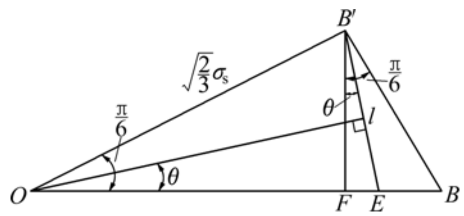


**Fig.3** Linear locus and polar angle on 30° sector on $\pi$-plane

Eq.(14) shows that the presentation of UY linear criterion on $\pi$-plane is a cluster right lines lying in the error triangle $FB'B$ made of TSS yield locus $B'B$ (hypotenuse) and Tresca's yield locus $B'F$ (right angled side), which starts from point $B'$ and aims at different points on $FB$. As criterion parameter $b$ changes from 0 to 1, the included angle $\theta$ is from 0 to π/6 covering all regimes from the lower bound (Tresca's) to the upper bound (TSS).

## 5 Application of Eq.(7)

In order to verify practicability and precision of Eq.(7), strip forging without bulge between two parallel platens is taken as an example. As shown in Fig.4. the top platen moves at a velocity, $-v_0$, while the bottom platen moves at a velocity, $v_0$. Assuming that the velocity component $v_y$ varies linearly with the $y$ coordinate, the

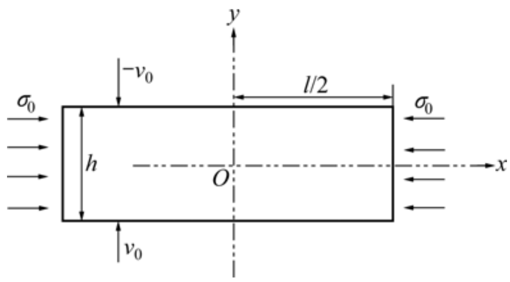


**Fig.4** Strip forging without bulge

kinematically admissible velocity and strain rate fields are respectively[15]

$$v_x=\frac{2v_0}{h}x,\ v_y=-\frac{2v_0}{h}y,\ v_z=0 \tag{15}$$

$$\dot{\varepsilon}_x=-\dot{\varepsilon}_y=\frac{2v_0}{h},\ \dot{\varepsilon}_z=0 \tag{16}$$

Above strain rates satisfy

$$\dot{\varepsilon}_{\max}=\dot{\varepsilon}_1=\dot{\varepsilon}_x=2v_0/h, \dot{\varepsilon}_{\min}=\dot{\varepsilon}_3=\dot{\varepsilon}_y=-2v_0/h,\ \dot{\varepsilon}_2=0 \tag{17}$$

Substituting Eq.(7) into following and noticing Eq.(17) yield

$$\dot{W}_i=\int_V D(\dot{\varepsilon}_{ij})\mathrm{d}V=\int_V\frac{1+b}{2+b}\sigma_s(\dot{\varepsilon}_{\max}-\dot{\varepsilon}_{\min})\mathrm{d}V=$$
$$2\frac{1+b}{2+b}\sigma_s\int_0^l\int_0^{h/2}\frac{4v_0}{h}\mathrm{d}y\mathrm{d}x=4\frac{1+b}{2+b}\sigma_s v_0 l \tag{18}$$

From Eq.(15), velocity discontinuity at interface is

$$\Delta v_f=v_x=\frac{2v_0}{h}x$$

Let friction stress is $\tau_f$=$mk$=$m\sigma_s/\sqrt{3}$, then

$$\dot{W}_f=mk\int_{F_f}\left|\Delta v_f\right|\mathrm{d}F=4mk\frac{2v_0}{h}\int_0^{l/2}x\mathrm{d}x=m\frac{l^2\sigma_s}{h\sqrt{3}}v_0 \tag{19}$$

Noticing Eq.(15), the power to overcome the resistance of external pressure $\sigma_0$ is

$$\dot{W}_b=4\frac{h}{2}v_x\sigma_0=2lv_0\sigma_0 \tag{20}$$

Summing Eqs.(18), (19) and (20) results in upper bound power:

$$J^*=\dot{W}_i+\dot{W}_f+\dot{W}_b=4\frac{1+b}{2+b}\sigma_s v_0 l+m\frac{l^2\sigma_s}{h\sqrt{3}}v_0+2lv_0\sigma_0$$

Let the applied power $J=2\overline{p}lv_0=J^*$, then

$$\overline{p}=2\frac{1+b}{2+b}\sigma_s+m\frac{l\sigma_s}{2h\sqrt{3}}+\sigma_0 \tag{21}$$

Eq.(16) belongs to plane strain, where[16], the parameter $b=1/(1+\sqrt{3})$. Substituting it into Eq.(21) and taking $k=\sigma_s/\sqrt{3}$, then rearranging yields

$$\frac{\bar{p}}{2k} = 1 + \frac{m}{4}\frac{l}{h} + \frac{\sigma_0}{2k} \tag{22}$$

It is obviously that Eq.(22) is the same result with upper bound solution deduced by AVITZUR[15] with Mises' criterion. When $m$=1, $\sigma_0$=0, it becomes

$$\frac{\bar{p}}{2k} = 1 + \frac{0.25l}{h} \tag{23}$$

Eq.(23) is also identical to that of strip forging solved by Slab method[17] with sticking friction without resistance of external pressure.

According to the universality of Eq.(7), we can also get specific linear plastic work rate per unit volume for plane stain by substitute parameter $b = 1/(1+\sqrt{3})$ [16] into Eq.(7):

$$D(\dot{\varepsilon}_{ij}) = (\dot{\varepsilon}_{\max} - \dot{\varepsilon}_{\min})\sigma_s/\sqrt{3} \tag{24}$$

Eq.(24) is specific linear plastic work rate per unit volume with $b = 1/(1+\sqrt{3})$. For plane strain, it is the same with plastic work rate per unit volume by Mises' criterion, because the yield locus corresponding to Eq.(24) is a inscribed dodecagon, with its apex just lying on Mises' yield locus[16], as shown in Fig.1. Therefore, integrating Eq.(24) yields directly also the result of Eq.(22). For other plane strain deformations, integrating Eq.(24) to get internal deformation power is also recommended.

## 6 Conclusions

1) The unified linear plastic work rate done per unit volume is first deduced for the UY criterion. It corresponds to various specific linear criteria according to suitable $b$ values.

2) The representation of UY yield criterion on the $\pi$-plane is a cluster right lines lying in the error triangle, which includes different angles with Tresca's yield locus and passes apex of Tresca's hexagon.

3) With the parallel velocity field, the unified linear plastic work rate per unit volume is first applied to analysis of strip forging and an analytical solution of average pressure is obtained.

4) For plane strain, taking $b = 1/(1+\sqrt{3})$ for Eq.(21) or integrating Eq.(24) all yields the same upper bound solution with that by Mises' criterion. When $m$=1, $\sigma_0$=0, it is the same with solution by Slab method.

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