Chapter 2

The basics of turbulence modelling

In this chapter, some basic concepts of turbulence modelling are discussed in order to introduce the inherent hypotheses as well as the necessary terminology.

2.1 The Navier-Stokes equations

The basic equations describing fluid dynamics are the Navier-Stokes equations. They express the conservation of mass, momentum and energy. These conservation laws can easily be derived by considering the flow in a volume V, fixed to the coordinate system in which all quantities are measured. Consider this volume to have a surface A_V. The velocity in each point is represented by the vector $\overline{u}$, the density by ρ. In each point of the surface A_V, the unit outward normal is called $\overline{n}$.

2.1.1 Mass conservation: continuity equation

Mass conservation means that the net mass which enters the control volume V, through the surface, can be found in the volume V:

$$\frac{\partial}{\partial t} \int_V \rho dV + \int_{A_V} \rho \overline{u}.\overline{n} dA_V = 0$$

As this equation is valid in any control volume, the continuity equation can be rewritten using the divergence theorem as:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \overline{u}) = 0 \tag{2.1}$$

Equation (2.1) is the differential form of the mass-conservation law and is also known as the continuity equation.

2.1.2 Momentum conservation

Following Newtons second law, the increase in momentum (both by unsteadyness and transport) during a time unit is equal to the sum of the forces working on the volume. Apart from the external forcefields (e.g. gravitational forces) summarized as the force $\overline{f}$ per unit of mass, a second force appears as a stress $\overline{T}$ exerted on an elementary surface dA_V. This stress can be coupled to a stress tensor $\overline{\overline{\sigma}}$ by expressing equilibrium as: $\overline{T} = \overline{n}.\overline{\overline{\sigma}}$. Because the local equilibrium implies $\overline{\overline{\sigma}}$ is symmetric, the stress tensor can be written as the sum of a pressure-term and a symmetric and traceless viscous shear stress tensor $\overline{\overline{\tau}}$: $\overline{\overline{\sigma}} = -p\overline{\overline{I}} + \overline{\overline{\tau}}$.
This means that Newtons second law, applied on the volume V results in:

$$\frac{\partial}{\partial t} \int_V \rho \overline{u} dV + \int_{A_V} \rho \overline{u} \; \overline{u}.\overline{n} dA_V = \int_V \rho \overline{f} dV + \int_{A_V} \overline{T} dA_V$$

This can be transformed into a differential formulation:

$$\frac{\partial \rho \overline{u}}{\partial t} + \nabla.(\rho \overline{u} \; \overline{u}) + \nabla p = \rho \overline{f} + \nabla.\overline{\overline{\tau}} \tag{2.2}$$

2.1.3 Energy conservation

The first law of thermodynamics states that the energy-increase (by both unsteadyness and convection) in the volume V is equal to the sum of the work exerted on the volume V and the supplied heat. Using the notation Q for the heat per unit of volume and time generated by chemical and physical processes and the notation $\overline{q}$ for the heat flux vector, this law can be translated into the equation:

$$\frac{\partial}{\partial t} \int_V \rho E dV + \int_{A_V} \rho E \overline{u}.\overline{n} dA_V =$$
$$\int_V \rho \overline{f}.\overline{u} dV + \int_{A_V} \overline{T}.\overline{u} dA_V - \int_{A_V} \overline{q}.\overline{n} dA_V + \int_V Q dV$$

where E is the specific total mechanical energy (sum of internal and kinetic energy $E = e + \frac{1}{2}\overline{u}.\overline{u}$). If only conservative mass forces ($\overline{f} = -\nabla U$) are present, their potential energy can be included in the energy E which allows writing the differential form of the energy equation as:

$$\frac{\partial \rho E}{\partial t} + \nabla.(\rho E\overline{u}) + \nabla.(p\overline{u}) = \nabla.(\overline{\overline{\tau}}.\overline{u}) - \nabla.\overline{q} + Q$$

Using the definition of specific total enthalpy $H = E + \frac{p}{\rho}$, this can be written in the form:

$$\frac{\partial \rho E}{\partial t} + \nabla.(\rho H\overline{u}) = \nabla.(\overline{\overline{\tau}}.\overline{u}) - \nabla.\overline{q} + Q \tag{2.3}$$

2.1.4   Introduction of shear and rotation concept

If the velocity in a certain point is given by $\overline{u}$, the velocity in a point at a distance $d\overline{x}$ of this point is: $\overline{u} + d\overline{x}.\nabla\overline{u}$. $\nabla\overline{u}$ can be seen as the sum of an anti-symmetric part $\overline{\overline{\Omega}}$ and a symmetric part $\overline{\overline{S}}$. $\overline{\overline{\Omega}}$ is called the rotation rate tensor, as it has the form of a rotation: $d\overline{x}.\overline{\overline{\Omega}} = \frac{1}{2}(\nabla \times \overline{u}) \times d\overline{x}$. The symmetric part $\overline{\overline{S}}$ of $\nabla\overline{u}$, called the strain rate (or shear) tensor, can be written as the sum of a diagonal tensor and a traceless tensor $\overline{\overline{\gamma}} = \overline{\overline{S}} - \frac{1}{3}\left\{\overline{\overline{S}}\right\}\overline{\overline{I}}$. The components ($S_{ij}$ and Ω_{ij}) of the strain rate (or shear) ($\overline{\overline{S}}$) and rotation ($\overline{\overline{\Omega}}$) tensors can be written as:

$$S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$
$$\Omega_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) \tag{2.4}$$

where u_i are the components of the velocity vector $\overline{u}$ and x_i the components of the position vector $\overline{x}$.

2.1.5   Constitutive relations

In order to transform the Navier-Stokes equations (2.1, 2.2, 2.3) into a closed system of equations, some additional relations, called constitutive relations are necessary. The constitutive relations for a Newtonian fluid are by definition:

$$\overline{\overline{\tau}} = 2\mu\overline{\overline{\gamma}} \qquad \text{Newtons law}$$
$$\overline{q} = -\kappa\nabla T \qquad \text{Fouriers law}$$

where μ is the dynamic viscosity and κ the heat transfer coefficient.

In cases where the fluid is a liquid, and no explicit heat transfer is present, the temperature can be considered constant because of the large specific heat of a liquid. This implies the energy-equation is not needed to obtain the flow-prediction.

If however, the fluid is an ideal gas, $e = e(T)$ and $p = \rho R T$. For a polytropic gas, the specific heat coefficients $c_v = \frac{de}{dT}$ and $c_p = \frac{dh}{dT}$ are constants, which means the energy can be written as: $e = c_v T$ or $h = c_p T$ Using the definition $\gamma = \frac{c_p}{c_v}$, the velocity of sound in such a gas is given by: $c = \sqrt{\frac{\gamma p}{\rho}}$. Using the previous definitions, along with the definition of the Prandtl number $Pr = \frac{\mu c_p}{\kappa}$, the Fourier law for a gas is usually written as

$$\overline{q} = -\frac{\mu}{Pr}\nabla h$$

2.1.6 Non-dimensionalization of the Navier-Stokes equations

For practical calculations, the Navier-Stokes equations are written in their dimensionless form. To obtain this form, a reference-length l_r, -pressure p_r and -density ρ_r are chosen. The reference-values for the remaining quantities are chosen such that the Navier-Stokes equations remain unchanged: $u_r^2 = p_r/\rho_r$, $e_r = u_r^2$ and $q_r = \rho_r u_r^3$. Using the definition of the Reynolds-number $Re_r = \frac{\rho_r v_r l_r}{\mu_r}$, the constitutive relations can then be rewritten as (the subscript dl meaning the dimensionless value):

$$
\begin{aligned}
\overline{\overline{\tau}}_{dl} &= \frac{2}{Re_r}\frac{\mu}{\mu_r}\overline{\overline{\gamma}}_{dl} \\
\overline{q}_{dl} &= -\frac{1}{Re_r Pr_r}\frac{\mu}{\mu_r}\nabla h_{dl}
\end{aligned}
$$

If, for a gas, the reference temperature $T_r = \frac{p_r}{R\rho_r} = \frac{1}{R}e_r$ is chosen, the following relations exist between the dimensionless quantities:

$$p_{dl} = \rho_{dl}T_{dl}, \; e_{dl} = \frac{1}{\gamma - 1}T_{dl}, \; h_{dl} = \frac{\gamma}{\gamma - 1}T_{dl}, \; c_{dl}^2 = \gamma T_{dl} \; .$$

2.2   Statistical approach of turbulence: RANS

As was mentionned in the introduction, turbulence is basically described by the Navier-Stokes equations. However, directly solving these equation (DNS), as well as solving the large scales and modelling the small scales (LES) are no realistic options for design purposes (estimations for the required computer storage and speed can be found in [16]). Moreover, from practical point of view, a mean value of the flow is often sufficient for design purposes. This means that if DNS or LES are used, the result is subsequently averaged statistically. In the Reynolds averaging of the Navier-Stokes equations (RANS), the equations are averaged statistically before solving them. The averaging time T needs to be sufficiently large compared to the turbulent time-scales, and also sufficiently small compared to the time-scales of the mean flow.

2.2.1   Averaging techniques

2.2.1a   Reynolds-averaging

The starting point is the averaging technique introduced by Reynolds [40]. For engineering purposes, the most appropriate form of Reynolds averaging is the time averaging procedure, which can most clearly be explained for stationary turbulence. For such a flow, the instantaneous velocity, $u_i(\overline{x}, t)$ can be written as the sum of a mean part $\overline{u_i}(\overline{x})$ and a fluctuating part $u_i'(\overline{x}, t)$:

$$u_i(\overline{x}, t) = \overline{u_i}(\overline{x}) + u_i'(\overline{x}, t)$$

where the mean velocity is defined as the time-averaged value:

$$\overline{u_i}(\overline{x}) = \frac{1}{T} \int_t^{t+T} u_i(\overline{x}, t) dt$$

T is an averaging time, which satisfies the previously mentioned conditions. This form of averaging is what is generally called Reynolds-averaging. A few usefull properties can be written: the Reynolds-average of a Reynolds-averaged value is again the same value

$$\overline{\overline{u_i}}(\overline{x}) = \frac{1}{T} \int_t^{t+T} \overline{u_i}(\overline{x}) dt = \overline{u_i}(\overline{x})$$

and the Reynolds-averaged value of the fluctuating part is zero:

$$\overline{u_i'} = \frac{1}{T} \int_t^{t+T} [u_i(\overline{x}, t) - \overline{u_i}(\overline{x})] dt = \overline{u_i}(\overline{x}) - \overline{\overline{u_i}}(\overline{x}) = 0$$

Applying this kind of averaging to the compressible mass equation results in

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho u_i}}{\partial x_i} + \frac{\partial \overline{\rho' u_i'}}{\partial x_i}$$

It can be seen that due to the type of averaging, a new variable $\overline{\rho' u_i'}$ appears if a compressible flow is considered. In an analogeous way, triple correlations involving ρ', u_i' and u_j' will appear in the momentum equations. When using this kind of averaging, additional closure relations would be necessary for these correlations.

2.2.1b   Favre-averaging

The time-averaged equations can be simplified significantly by using the density-weighted averaging procedure suggested by Favre [64]. The Favre-averaging technique consists of introducing a mass-averaged velocity $\tilde{u}_i$, defined by

$$\tilde{u}_i = \frac{1}{\overline{\rho}} \frac{1}{T} \int_t^{t+T} \rho(\overline{x}, t) u_i(\overline{x}, t) dt \tag{2.5}$$

where $\overline{\rho}$ is the conventional Reynolds-averaged density. The velocity can now be written as

$$u_i(\overline{x}, t) = \tilde{u}_i + u_i''$$

Thus, rewriting (2.5) in terms of Reynolds-averaging, $\overline{\rho} \tilde{u}_i = \overline{\rho u_i} = \overline{\rho} \, \overline{u_i} + \overline{\rho' u_i'}$, it can be seen that in terms of Favre-averaging, conservation of mass can be rewritten as

$$0 = \frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho} \tilde{u}_i}{\partial x_i} \tag{2.6}$$

which formaly looks just the same as the laminar mass-conservation equation. Before deriving the Favre-averaged equations, a few usefull relations concerning the Favre-averaging, as well as the relation with Reynolds-averaging are summarized below.

$$\overline{\rho u_i''} = 0; \quad \overline{u_i''} \neq 0$$

$$\tilde{u}_i = \overline{u_i} + \frac{\overline{\rho' u_i'}}{\overline{\rho}}$$

$$\overline{u_i''} = -\frac{\overline{\rho' u_i''}}{\overline{\rho}} = -\frac{\overline{\rho' u_i'}}{\overline{\rho}}$$

$$
\begin{aligned}
\overline{\rho \phi \psi} &= \overline{\rho(\tilde{\phi} + \phi'')(\tilde{\psi} + \psi'')} \\
&= \overline{\rho(\tilde{\phi}\tilde{\psi} + \phi''\tilde{\psi} + \tilde{\phi}\psi'' + \phi''\psi'')} \\
&= \overline{\rho}\tilde{\phi}\tilde{\psi} + \overline{\rho\phi''\psi''}
\end{aligned}
$$

2.2.2   The averaged equations

2.2.2a   Mass-equation

In section 2.2.1, the averaged form (2.6) of the mass-conservation equation was obtained by combining Reynolds-averaging (used for ρ and p) and Favre-averaging (remaining quantities). This equation contains no extra terms compared to the initial mass-equation, it is said the equation is of a closed form. Unfortunately, applying Favre-averaging to the momentum- and energy equations does not result in a closed formulation but in an open formulation. This means that compared to the original equations, additional terms will appear in the averaged equations. These terms will need to be modelled. This need for modelling is referred to as the closure problem.

2.2.2b   Momentum-equation

If no external forces are present, the original momentum-equation, as derived in 2.1.2 can be summarized as:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

The stresses acting on a surface can be defined as:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} = -p\delta_{ij} + 2\mu(S_{ij} - \frac{1}{3}\delta_{ij}S_{ii})$$

where S_{ij} is the shear tensor, which was defined by equation (2.4).

Applying the averaging techniques to the momentum-equation results in:

$$\frac{\partial \overline{\rho} \tilde{u}_i}{\partial t} + \frac{\partial \overline{\rho} \tilde{u}_i \tilde{u}_j}{\partial x_j} = -\frac{\partial \overline{\rho u_i'' u_j''}}{\partial x_j} - \frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{2\mu(S_{ij} - \frac{1}{3}\delta_{ij} S_{ii})}}{\partial x_j} \tag{2.7}$$

The additional term $-\overline{\rho u_i'' u_j''}$ appears as a consequence of the non-linearity of the convective terms. As mentioned before, this term needs modelling (the closure problem).

2.2.2c   Energy equation

In the absence of external heat Q, the original energy-equation, as derived in 2.1.3 can be summarized as:

$$\frac{\partial \rho E}{\partial t} + \frac{\partial \rho H u_j}{\partial x_j} = \frac{\partial \tau_{ij} u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j}$$

while the averaged form of this equation can be written as (see for example [16, 33]):

$$\frac{\partial \overline{\rho} \tilde{E}}{\partial t} + \frac{\partial \overline{\rho} \tilde{u}_j \tilde{H}}{\partial x_j} = \frac{\partial}{\partial x_j} \left[-\overline{q_j} - \overline{\rho u_j'' h''} + \overline{\tau_{ij}'' u_i''} - \overline{\rho u_j'' \frac{1}{2} u_i'' u_i''} \right]$$
$$+ \frac{\partial}{\partial x_j} \left[\tilde{u}_i (\overline{\tau_{ij}} - \overline{\rho u_i'' u_j''}) \right] \tag{2.8}$$

2.2.2d   The RANS equations

The set of equations (2.6), (2.7) and (2.8) is called the Reynolds averaged Navier Stokes equations (RANS). These RANS-equations express the transport of the mean quantities. In order to be able to solve these equations, additional equations are needed for the six components (3D) of the symmetric Reynolds-stress tensor $-\overline{\rho u_i'' u_j''}$ as well as for the three components of the turbulent heatflux vector $-\overline{\rho u_i'' h}$.

2.2.3   Exact Reynolds-stress transport equations

In order to obtain the exact transport equations for the Reynolds-stresses, the Navier-Stokes equations are multiplied by a fluctuating quantity. The resulting equations are then combined to differential equations which result in equations for the Reynolds-stresses after averaging. If the mass-equation is abbreviated as $Ma(\rho) = 0$ and the momentum-equation by $Mo(u_i) = 0$, the Reynolds-stress equations can be obtained by the averaging $\overline{u_i'' Mo(u_j) + u_j'' Mo(u_i) + u_i'' u_j'' Ma(\rho)} = 0$.

The equations resulting from this procedure can be written schematically as [16, 33]:

$$\frac{\partial \overline{\rho u_i'' u_j''}}{\partial t} + \frac{\partial}{\partial x_k}\left[\overline{\rho u_i'' u_j''}\tilde{u}_k\right] = P_{ij} - \overline{\rho}\epsilon_{ij} - \frac{\partial J_{ijk}}{\partial x_k} + \Pi_{ij} - \overline{u_j''}\frac{\partial \overline{p}}{\partial x_i} - \overline{u_i''}\frac{\partial \overline{p}}{\partial x_j} \qquad (2.9)$$

where

$$
\begin{aligned}
P_{ij} &= -\left[\overline{\rho u_j'' u_k''}\frac{\partial \tilde{u}_i}{\partial x_k} + \overline{\rho u_i'' u_k''}\frac{\partial \tilde{u}_j}{\partial x_k}\right] \\
J_{ijk} &= \overline{\rho u_i'' u_j'' u_k''} + \delta_{ik}\overline{p' u_j''} + \delta_{jk}\overline{p' u_i''} - \overline{u_j'' \tau_{ik}} - \overline{u_i'' \tau_{jk}} \\
\Pi_{ij} &= \overline{p'\left(\frac{\partial u_j''}{\partial x_i} + \frac{\partial u_i''}{\partial x_j}\right)} \\
\overline{\rho}\epsilon_{ij} &= \overline{\tau_{ik}\frac{\partial u_j''}{\partial x_k}} + \overline{\tau_{jk}\frac{\partial u_i''}{\partial x_k}}
\end{aligned} \qquad (2.10)
$$

The physical meaning of these terms can be described as follows.

Production P_{ij}

Turbulent stresses are generated at the expense of mean flow energy by mean flow deformation. This term does not need any closure.

Transport J_{ijk}

This term consists of several parts.

First, there is the turbulent diffusion term (transport through velocity fluctuations) which is given by $\overline{\rho u_i'' u_j'' u_k''}$. A closure model is necessary for this term.

A second part of the transport term is the pressure transport $\delta_{ik}\overline{p' u_j''} + \delta_{jk}\overline{p' u_i''}$. This term also needs closure.

The remaining term in the transport term is called the molecular diffusion term, which, for low compressibility can be written as: $-\overline{u_j'' \tau_{ik}} - \overline{u_i'' \tau_{jk}} = \mu\frac{\partial}{\partial x_k}\left(\widetilde{u_i'' u_j''}\right)$

Pressure-strain term Π_{ij}

Pressure fluctuations redistribute the turbulent stress among components to make turbulence more isotropic. The pressure straining effects can be considered to exist of four components, which appear when Π_{ij} is rewritten using the exact Poisson equation for the pressure fluctuations [65]. The first is called the "slow term" and

represents the return to isotropy of non-isotropic turbulence: in the absence of mean rate of strain S_{ij} and a body force, away from a solid wall, pressure fluctuations force turbulence to approach an isotropic state. The second part of the pressure-strain term is called the "rapid term". This term represents the isotropization of the process of stress production due to S_{ij}. Pressure fluctuations slow down a preferential feeding of turbulence by S_{ij} into a selected direction. The third part of the pressure-strain term is the isotropization of stress production due to a body force, while the last term is the wall pressure reflection term, the "splatting" effect. All of these parts of the pressure-strain term need closure.

Stress dissipation ϵ_{ij}

The term $\epsilon_{ij} = \frac{1}{\rho}\left[\overline{\tau_{ik}\frac{\partial u_j''}{\partial x_k}} + \overline{\tau_{jk}\frac{\partial u_i''}{\partial x_k}}\right] = 2\nu\overline{\frac{\partial u_i''}{\partial x_k}\frac{\partial u_j''}{\partial x_k}}$ represents the stress dissipation which mainly occurs at the smallest scales.

2.2.4 Differential Reynolds-stress models (RSM)

In section 2.2.3, the exact equations for the Reynolds-stresses were discussed. In order to be able to solve these equations, closure is necessary for the transport, pressure-strain and dissipation terms. One could again apply the same technique as the one which led to the Reynolds-stress equations in order to obtain the equations for the necessary higher order correlations, however this would not be very usefull as new correlations would appear which would again need closure.

In differential Reynolds-stress models (RSM), the closure of the RANS equations is achieved by introducing models for the transport, pressure-strain and dissipation terms in the Reynolds-stress equations. This means that from now on, the exact RANS will not be used anymore, but a turbulence model, which approximates these equations is introduced.

In the following, some examples of the modelling used for the terms which need closure are given. For the more advanced modelling, the appropriate literature should be consulted (some examples can be found in [63, 66])

2.2.4a Transport term J_{ijk}

The simplest approach consists of applying the gradient diffusion hypothesis (Daly & Harlow [67]) to model the turbulent velocity diffusion as:

$$\frac{\partial}{\partial x_k}\overline{\rho u_i'' u_j'' u_k''} = \frac{\partial}{\partial x_k}\left(c_s\frac{k}{\epsilon}\widetilde{u_k'' u_m''}\frac{\partial \widetilde{u_i'' u_j''}}{\partial x_m}\right)$$

where $c_s = 0.22$ is a common value.

The pressure transport has a different nature (propagation of disturbances) and the gradient diffusion hypothesis is not applicable for this term. Yet, it is common to "lump" this transport with the turbulent velocity diffusion (c_s should account for this).

2.2.4b   Pressure-strain term Π_{ij}

Various models have been proposed for this term since the first attempt by Rotta [49]. No expressions are given here, examples are the models of Lumley [68], Hanjalic [66], Launder & Tselipidakis [69] and Launder & Li [70].

2.2.4c   Dissipation term ϵ_{ij}

At high Reynolds numbers, the large scale motion is unaffected by viscosity, while the fine-scale structure is locally isotropic (unaware of the large eddies' orientations). Consequently, the correlation for ϵ_{ij} which is associated with the smallest eddies, should reduce to zero if $i \neq j$, while for $i = j$, all three components should be equal. Hence, a common way to model the viscous destruction of stresses for high-Reynolds number flows is:

$$\epsilon_{ij} = \frac{2}{3}\delta_{ij}\epsilon \tag{2.11}$$

where $\epsilon = \nu \overline{\frac{\partial u_l''}{\partial x_k}\frac{\partial u_l''}{\partial x_k}}$. In order to determine the value of ϵ, an additional transport equation is used. Just as the Reynolds-stress transport equations, the exact ϵ-equation can be derived by manipulation of the RANS equations. After elaborate manipulation, the following transport equation for the dissipation rate ϵ can be written (the averaging signs $\bar{}$ and $\tilde{}$ are omitted for simplicity):

$$\frac{D\epsilon}{Dt} = P^{\epsilon(1)} + P^{\epsilon(2)} + P^{\epsilon(3)} + T^\epsilon - D^\epsilon - \frac{\partial}{\partial x_m}\left(J_m^\epsilon - \nu\frac{\partial \epsilon}{\partial x_m}\right) \tag{2.12}$$

where the various terms have been grouped in the standard way:

$$P^{\epsilon(1)} = -2\nu S_{ij}\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}$$
$$P^{\epsilon(2)} = -2\nu S_{ij}\frac{\partial u_k'}{\partial x_i}\frac{\partial u_k'}{\partial x_j}$$

$$
\begin{aligned}
P^{\epsilon(3)} &= -2\nu \frac{\partial^2 u_i}{\partial x_j \partial x_k} u'_j \frac{\partial u'_i}{\partial x_k} \\
T^{\epsilon} &= -2\nu \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k} \\
D^{\epsilon} &= 2\nu^2 \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \\
J^{\epsilon}_m &= \nu \left\{ \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} u'_m + \frac{2}{\rho} \frac{\partial p'}{\partial x_i} \frac{\partial u'_m}{\partial x_i} \right\}
\end{aligned}
$$

The physical interpretation of the terms is mean flow related production ($P^{\epsilon(1)}$, $P^{\epsilon(2)}$ and $P^{\epsilon(3)}$), turbulence related production through vortex stretching (T^{ϵ}), viscous destruction (D^{ϵ}), turbulent transport flux (J^{ϵ}_m) and viscous diffusive flux. All terms in this equation, except for the viscous diffusive flux term, need to be modelled.

2.2.5 Algebraic Stress Models (ASM)

A considerable simplification compared to RSM can be obtained if instead of writing six transport equations for the turbulent stresses, only one transport equation is written for the kinetic energy of the turbulence $k = \frac{1}{2} \widetilde{u''_i u''_i}$. The exact equation for k can be obtained by determining the trace of the equations which determine the Reynolds-stress tensor. The resulting equation can look like:

$$
\frac{\partial \overline{\rho} k}{\partial t} + \frac{\partial}{\partial x_k} \left[\overline{\rho} \tilde{u}_k k + \overline{\rho k u''_k} + \overline{p' u''_k} - \overline{u''_i \tau_{ik}} \right]
$$
$$
= -\overline{\rho u''_i u''_k} \frac{\partial \tilde{u}_i}{\partial x_k} + \overline{p' \frac{\partial u''_i}{\partial x''_i}} - \overline{u''_i} \frac{\partial \overline{p}}{\partial x_i} - \overline{\tau_{ik} \frac{\partial u''_i}{\partial x_k}}
$$

In a turbulence modelling context, the standard way of grouping the terms in this equation is as follows:

$$
\frac{\partial \overline{\rho} k}{\partial t} + \frac{\partial}{\partial x_k} (\overline{\rho} \tilde{u}_k k) = P_k - \overline{\rho} \epsilon - \frac{\partial}{\partial x_k} \left(J_k - \mu \frac{\partial k}{\partial x_k} \right) \tag{2.13}
$$

where

$$
\begin{aligned}
P_k &= -\overline{\rho \widetilde{u''_i u''_k}} \frac{\partial \tilde{u}_i}{\partial x_k} \\
\epsilon &= \overline{\tau_{ik} \frac{\partial u''_i}{\partial x_k}} = \overline{\frac{\partial u''_i}{\partial x_k} \frac{\partial u''_i}{\partial x_k}}
\end{aligned}
$$

$$J_k \;\; = \;\; \overline{\rho} \frac{1}{2} \overline{u_i'' u_i'' u_k''} + \overline{u_k'' p'}$$

P_k represents production of turbulent kinetic energy, i.e. transfer of energy from the mean flow to the turbulent fluctuations. The production term is positive in most cases, but can, under some conditions, temporarily and locally become negative.

Just as in the RSM modelling, closure of this set of equations (RANS and k-equation) necessitates modelling, in this case of ϵ, J_k and the Reynolds-stresses. An additional transport equation is used for ϵ, while J_k is usually related to strains and Reynolds-stresses. This means that, in order to obtain a closed formulation, the Reynolds-stresses remain to be modelled.

The algebraic stress modelling (ASM) approach of Rodi [71] consists of removing the Reynolds-stress advection and diffusion terms in the Reynolds-stress equations.

First, the anisotropy-tensor components need to be defined as

$$b_{ij} = \frac{\overline{u_i' u_j'}}{2k} - \frac{1}{3}\delta_{ij} \tag{2.14}$$

If an equilibrium can be considered for which convective and transport terms can be neglected (as e.g. homogeneous shear flow and the logarithmic region of an equilibrium boundary layer), the following constraint for the anisotropy components can be written:

$$\frac{Db_{ij}}{Dt} = 0$$

Using the definition of the anisotropy-components, this can be reformulated as:

$$\frac{D}{Dt}(\overline{u_i' u_j'}) = \frac{\overline{u_i' u_j'}}{k} \frac{Dk}{Dt} \tag{2.15}$$

Neglecting convective and transport terms in the k-equation (2.13) leads to the relation:

$$\frac{Dk}{Dt} = P_k - \rho\epsilon \tag{2.16}$$

Relations (2.15) and (2.16) can be combined to

$$\frac{D}{Dt}(\overline{u_i' u_j'}) = \frac{\overline{u_i' u_j'}}{k}(P_k - \rho\epsilon) \tag{2.17}$$

Finally, introducing (2.17) into the RSM equations (2.9), where the same assumptions (neglecting advection and diffusion) have been made and where (2.11) has been introduced leads to:

$$(P_k - \rho\epsilon)\frac{\overline{u_i' u_j'}}{k} = P_{ij} - \frac{2}{3}\epsilon\delta_{ij} + \Pi_{ij} \qquad (2.18)$$

This last relation represents the general form of an algebraic Reynolds-stress model. The differential equations for the Reynolds-stresses appearing in a RSM model have been reduced to the algebraic relation (2.18). The algebraic equivalent of a certain RSM model can be obtained by introducing the model for the pressure-strain term occuring in this RSM-model into relation (2.18). The resulting algebraic equation is implicit. This can be seen if the general form of the common models for Π_{ij} is considered:

$$\Pi_{ij} = \epsilon A_{ij}(\overline{\overline{b}}) + k\, M_{ijkl}(\overline{\overline{b}})\frac{\partial \overline{u_k}}{\partial x_l}$$

The most commonly used models are based upon a linear model of the form (Launder *et al.* [72]):

$$\frac{\pi}{\epsilon} = -c_1\overline{\overline{b}} + c_{01}\overline{\overline{S}} + c_{11}(\overline{\overline{b}}\,\overline{\overline{S}} + \overline{\overline{S}}\,\overline{\overline{b}} - \frac{2}{3}\{\overline{\overline{b}}\,\overline{\overline{S}}\}) + c_{12}(\overline{\overline{b}}\,\overline{\overline{\Omega}} + \overline{\overline{\Omega}}\,\overline{\overline{b}})$$

The algebraic system (2.18) can then be rewritten in the form

$$\overline{\overline{b}} = -\alpha\overline{\overline{S}} - \beta(\overline{\overline{b}}\,\overline{\overline{S}} + \overline{\overline{S}}\,\overline{\overline{b}} - \frac{2}{3}\{\overline{\overline{b}}\,\overline{\overline{S}}\}) - \gamma((\overline{\overline{b}}\,\overline{\overline{\Omega}} + \overline{\overline{\Omega}}\,\overline{\overline{b}}) \qquad (2.19)$$

which is clearly an implicit equation.

The algebraic relation describing the Reynolds-stresses is combined with a modelled form of the k-equation (2.13), and a modelled form of the ϵ-equation. This kind of modelling can be classified as two-equation turbulence modelling, as apart from the RANS equations, only two additional differential equations (for k and ϵ) appear.

A main drawback of the implicit ARSM approach is the fact that a strongly coupled non-linear set of algebraic equations has to be solved numerically, at each time-step. One feature of algebraic relations is the lack of damping or diffusion. For general complex flow situations, this often causes numerical problems in terms of stability and slow convergence. The computational effort sometimes becomes even larger than that for a full RSM.

2.2.6   Explicit algebraic Reynolds-stress models (EARSM) and non-linear eddy-viscosity turbulence models

The numerical difficulties associated with implicit ASM have motivated efforts to find explicit forms of the relation between the Reynolds-stress anisotropy-tensor and the mean flow quantities. The explicit form of this relation is then used together with a two-equation model like e.g. the k-ϵ model.

2.2.6a   Explicit algebraic Reynolds-stress models (EARSM)

The idea of generating an explicit algebraic Reynolds-stress model (EARSM) from an implicit ASM was first described by Pope [20], and later further developed by Gatski and Speziale [4]. It starts from the observation that the solution of equation (2.19) is of the general form

$$\overline{\overline{b}} = f(\overline{\overline{S}}, \overline{\overline{\Omega}}) \tag{2.20}$$

It can be shown that imposing invariance under an orthogonal coordinate transformation leads to the following required form for $\overline{\overline{b}}$:

$$\overline{\overline{b}} = \sum_i G^{(i)} T^{(i)} \tag{2.21}$$

where $T^{(i)}$ is the integrity basis for functions of a symmetric and asymmetric tensor and $G^{(i)}$ are scalar functions of the irreducible invariants of these tensors. For the case under consideration, the integrity basis consists of terms like e.g. $T^{(1)} = \overline{\overline{S}}$, $T^{(2)} = \overline{\overline{S}}\,\overline{\overline{\Omega}} - \overline{\overline{\Omega}}\,\overline{\overline{S}}$, ... while $\eta_1 = \{\overline{\overline{S}}^2\}$ is an example of an invariant (see Spencer [73]; Pope [20]).

By substitution of (2.21) into (2.19), Gatski and Speziale [4] obtained an explicit (exact) solution of an algebraic stress model in terms of $\overline{\overline{S}}$ and $\overline{\overline{\Omega}}$. However, in the resulting relation, the denominator of the coefficients $G^{(i)}$ contains a sum of positive and negative terms which has the potential to become zero, rendering singular behaviour, which could occur in complex non-equilibrium flows (whenever large strains occur). Hence the need to regularize these explicit algebraic stress models is clear. The regularization procedure proposed in [4] consisted of a Pade approximation, leading to a non-singular behaviour.

A somewhat different approach was recently proposed by Apsley and Leschziner [13]. Instead of attempting to solve the algebraic system exactly, they applied a repeated iterative approximation, which can be shortly described as follows. A first approximation for (2.19) is given by $\overline{\overline{b}}^{(1)} = -\alpha \overline{\overline{S}}$. Introducing $\overline{\overline{b}}^{(1)}$ into the right-hand side of

(2.19), leads to a new approximation $\overline{\overline{b}}^{(2)}$ which is quadratic, while a cubic approximation $\overline{\overline{b}}^{(3)}$ is obtained by introducing $\overline{\overline{b}}^{(2)}$ into the right-hand side of(2.19).

In the resulting EARSM, the coefficients resulting from the initial RSM could be used, however, this was not done in [13], instead, the relations between the coefficients were maintained, while the values for the coefficients were calibrated to return the correct behaviour of all stresses in a shear flow.

2.2.6b   Non-linear eddy-viscosity turbulence models

From the above explanation, it is seen that the model developed by Apsley and Leschziner [13], although it was developed using some concept of EARSM, is not directly linked to an existing RSM as some coefficients in the resulting constitutive relation were calibrated independently. This means that this model should be classified as a non-linear eddy-viscosity model instead of a EARSM.

When applying non-linear eddy-viscosity turbulence modelling, a relation of the form (2.21) is written, but instead of introducing this relation into an ASM, as in the EARSM approach, attempts are made to propose a form for the scalar functions $G^{(i)}$ which allow good anisotropy-predictions for a few simple flows.

The general form of a constitutive relation of third order can be shown to have the form (see appendix A)

$$
\begin{aligned}
2b_{ij} = \quad & \frac{\overline{u_i' u_j'}}{k} - \frac{2}{3}\delta_{ij} \\
= \quad & -2c_\mu\left(\tilde{S}_{ij} - \frac{1}{3}\delta_{ij}\tilde{S}_{ll}\right) \\
& +c_1\left(\tilde{S}_{ik}\tilde{S}_{kj} - \frac{1}{3}\delta_{ij}\tilde{S}_{lk}\tilde{S}_{kl}\right) + c_2\left(\tilde{\Omega}_{ik}\tilde{S}_{kj} - \tilde{S}_{ik}\tilde{\Omega}_{kj}\right) \\
& +c_3\left(\tilde{\Omega}_{ik}\tilde{\Omega}_{kj} - \frac{1}{3}\delta_{ij}\tilde{\Omega}_{lk}\tilde{\Omega}_{kl}\right) \\
& +c_4\left(\tilde{\Omega}_{ik}\tilde{S}_{kl}\tilde{S}_{lj} - \tilde{S}_{ik}\tilde{S}_{kl}\tilde{\Omega}_{lj}\right) \\
& +c_5\left(\tilde{\Omega}_{ik}\tilde{\Omega}_{kl}\tilde{S}_{lj} + \tilde{S}_{ik}\tilde{\Omega}_{kl}\tilde{\Omega}_{lj} - \tilde{\Omega}_{lk}\tilde{\Omega}_{kl}\tilde{S}_{ij} - \frac{2}{3}\tilde{\Omega}_{kl}\tilde{S}_{lm}\tilde{\Omega}_{mk}\delta_{ij}\right) \\
& +c_6\left(\tilde{S}_{lk}\tilde{S}_{kl}\tilde{S}_{ij}\right) + c_7\left(\tilde{\Omega}_{lk}\tilde{\Omega}_{kl}\tilde{S}_{ij}\right)
\end{aligned}
\tag{2.22}
$$

where the coefficients c_i are functions of the invariants. $\tilde{S}_{ij}$ and $\tilde{\Omega}_{ij}$ are the dimensionless shear and rotation components: $\tilde{S}_{ij} = \tau S_{ij}$ and $\tilde{\Omega}_{ij} = \tau\Omega_{ij}$ where τ is a turbulent time-scale.

Examples of this kind of modelling include the quadratic models of Shih *et al.* [5, 6, 51, 55], Khodak and Hirsch [1] and the cubic models of Craft *et al.* [10] ,Lien *et al.* [12] and Apsley and Leschziner [13].

It is this kind of approach (non-linear eddy-viscosity modelling) which is used in this work.

2.2.7   Linear two-equation turbulence models

Two-equation models discussed here are based on the Boussinesq hypothesis, which is similar to the Newtonian fluid hypothesis. In this hypothesis, a turbulent viscosity ν_t, depending on flow related features, connects the shear components to the Reynolds-stresses in the same way as the molecular viscosity, which is a material property, connects the shear components to the stress in Newtons hypothesis. The Boussinesq hypothesis can thus be written as:

$$\tau_{ij}^R = \overline{-\rho u_i'' u_j''} = 2\mu_t \left(S_{ij} - \frac{1}{3}\delta_{ij} S_{ll} \right) - \frac{2}{3}\rho k \delta_{ij} \tag{2.23}$$

A closer look at this relation shows us that the Boussinesq hypothesis is of the same form as the first order term in a non-linear relation (see equation (2.22)). The last term in (2.23) is guaranteeing that the trace of τ_{ij}^R is $-2\rho k$.
For two-equation models, the modelled form of the k-equation can be written as

$$\frac{Dk}{Dt} = 2\nu_t S_{ij} S_{ij} - \epsilon + \frac{\partial}{\partial x_i} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_i} \right] \tag{2.24}$$

The modelled form of the production term is simply a consequence of the Boussinesq hypothesis, while the flux term is modelled by a gradient diffusion expression (σ_k is a modelling coefficient).
The length scale, characterizing the size of the large, energy-containing eddies is subject to transport processes in a manner similar to the energy k. Therefore, this length scale is determined by introducing an additional differential equation. If ϵ is chosen as the second quantity which is determined by a differential equation, no further modelling is necessary. If, as in some models, an alternative quantity is determined by a differential equation, the dissipation term needs to be modelled in terms of this quantity and k.
A length scale equation does not necessarily need the length scale itself as dependent variable. Any combination like $Z = k^m l^n$ can be used, because k is known by solving the k-equation. Most equations so far do not use l as a variable. The most frequently used variables are the dissipation $\epsilon \sim \frac{k^{3/2}}{l}$, and the specific dissipation rate $\omega \sim \frac{\epsilon}{k} \sim \frac{k^{1/2}}{l}$. The standard structure of the transport equation for Z (at high Reynolds number) is:

$$\frac{DZ}{Dt} = c_{Z1}\frac{Z}{k}P_k - c_{Z2}\frac{Z}{k}\epsilon + \frac{\partial}{\partial x_i} \left[\frac{\nu_t}{\sigma_Z} \frac{\partial Z}{\partial x_i} \right] + Source$$

where P_k denotes the production term in the k-equation, c_{Z1} and c_{Z2} are constants and the form of the possible source term depends on the choice of Z. The Z-equation can

be the result of two possible approaches. The first approach consists of constructing a Z-equation in an ad-hoc manner, trying to mimic some of the physics believed to be essential. The second approach consists of deriving the exact transport equation for Z, and subsequently trying to model the specific terms. Both approaches usually lead to similar results. The modelling of the Z-equation is still one of the major weaknesses in both two-equation and Reynolds-stress models.

2.2.7a The k-ϵ model

The choice $Z = \epsilon$ is by far the most popular choice for the length scale determining parameter. The model which is nowadays called the standard k-ϵ turbulence model, was developed in 1972 by Jones and Launder [74]. In this model, the RANS-equations are used together with the k-equation (2.24) and the following modelled form of the ϵ equation:

$$\frac{D\epsilon}{Dt} = c_{\epsilon 1} \frac{\epsilon}{k} P_k - c_{\epsilon 2} \frac{\epsilon^2}{k} + \frac{\partial}{\partial x_i} \left[\frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_i} \right] \qquad (2.25)$$

The eddy-viscosity in this model is taken to be

$$\nu_t = c_\mu \frac{k^2}{\epsilon}$$

The standard values for the model parameters are:

$$c_\mu = 0.09, \ \sigma_k = 1.0, \ \sigma_\epsilon = 1.3, \ c_{\epsilon 1} = 1.44, \ c_{\epsilon 2} = 1.92$$

A traditional way of deducing the above value for c_μ is to consider a thin shear flow with approximate balance between production and dissipation. With y as cross-stream coordinate, this results in

$$c_\mu = \nu_t \frac{\epsilon}{k^2} = \nu_t \frac{\nu_t \left(\frac{\partial u}{\partial y} \right)^2}{k^2} = \left(\frac{-\overline{u'v'}}{k} \right)^2$$

For thin shear flows $\left(\frac{-\overline{u'v'}}{k} \right) \approx 0.3$, resulting in $c_\mu = 0.09$. The deduction of the other parameters' values follows in chapter 5.
At this point, it should be emphasized that the standard k-ϵ model is a high-Reynolds model: near-wall treatment, low-Reynolds number formulation and boundary conditions will be discussed later.

2.2.7b   The k-ω model

Although the choice $Z = \omega$ can be traced back to Kolmogorov [48], the present popularity of the k-ω model is mainly due to the work done by Wilcox and co-workers (extensively presented in [33]). The proponents of the k-ω model claim that the main advantage of this model as compared to the k-ϵ model, lies in a more natural treatment of the near-wall region.

The quantity ω can be seen as an inverse time-scale of the large eddies. The dissipation is modelled as

$$\epsilon = c_\mu \omega k \tag{2.26}$$

In the model described by Wilcox [75, 33], the RANS equations are used together with the k-equation (2.24) and the following ω-equation:

$$\frac{D\omega}{Dt} = 2\alpha S_{ij} S_{ij} - \beta \omega^2 + \frac{\partial}{\partial x_i} \left[\left(\nu + \frac{\nu_t}{\sigma_\omega} \right) \frac{\partial \omega}{\partial x_i} \right] \tag{2.27}$$

together with the eddy-viscosity relation

$$\nu_t = \frac{k}{\omega}$$

The model parameter values are (Wilcox [33]):

$$c_\mu = 0.09, \ \sigma_k = 2.0, \ \sigma_\omega = 2.0, \ \alpha = \frac{5}{9}, \ \beta = \frac{3}{40}$$

2.2.8   One-equation and algebraic models

During the development of turbulence modelling, less complex, but less accurate models have been used. Examples include one-equation models, where only the modelled kinetic energy equation was added to the RANS-equations, while the length scale was defined geometrically. An even further approximation consists of algebraic modelling, where the RANS-equations were used with an algebraic prescription for the turbulent viscosity (and thus no additional transport equations). These models are not dicussed here.

2.3   Near-wall aspects

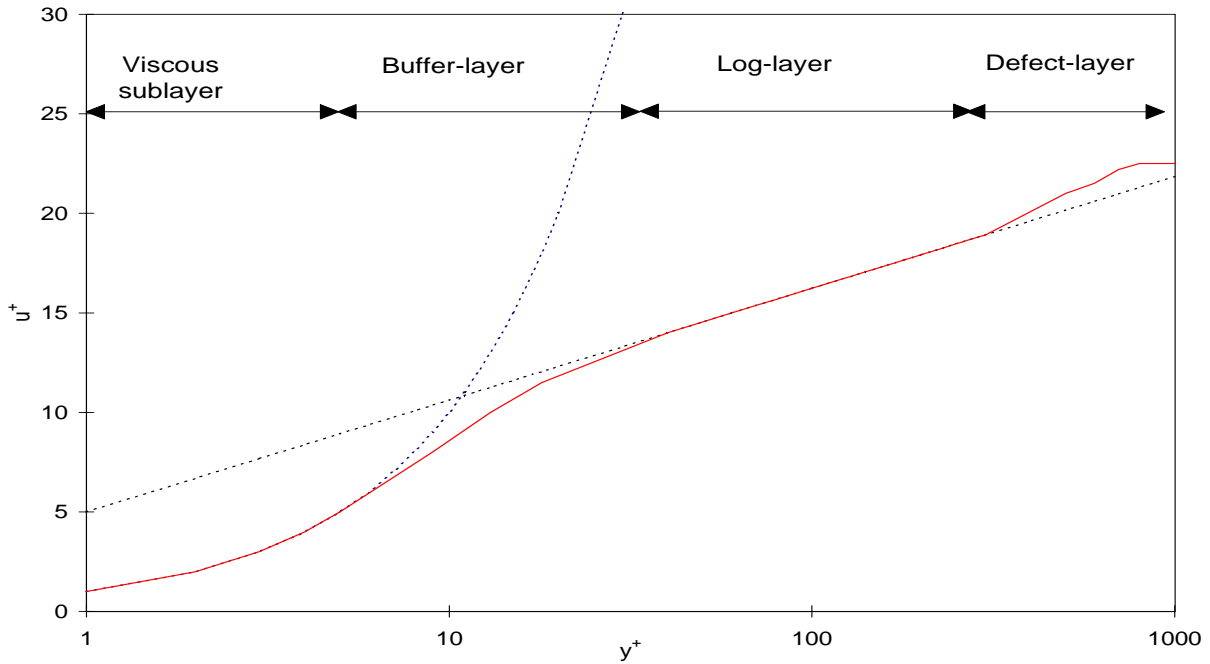### 2.3.1   Near-wall asymptotics

Figure 2.1: Velocity profile for a turbulent boundary layer.

Figure 2.1 shows a typical velocity profile for a turbulent boundary layer. The quantity y^+ is a dimensionless distance from the wall, while U^+ is a dimensionless velocity. Both quantities are defined by:

$$U^+ = \frac{u}{u_\tau}$$
$$y^+ = \frac{y u_\tau}{\nu} \tag{2.28}$$

where u_τ is the friction velocity, defined by $\tau_w = \rho u_\tau^2$, where τ_w is the wall shear stress. On figure 2.1, distinct regions in the velocity profile are denoted as the viscous sublayer, the buffer-layer, the log-layer and the defect layer.

The log-layer is by definition the part of the boundary layer sufficiently close to the surface so that inertial terms can be neglected, yet sufficiently distant so that the molecular, or viscous, stress is negligible compared to the Reynolds-stress. This region, where the law of the wall applies (logarithmic velocity profiles, see later), typically lies between $y^+ = 30$ and $y/\delta = 0.1$ ($\delta =$ boundary layer thickness), where the upper boundary is dependent upon Reynolds number.

The viscous sublayer is the region near the surface. In this region the velocity varies approximately linearly with y^+.

In the buffer-layer, the velocity-profile gradually asymptotes to the law of the wall for large values of y^+.

The defect layer lies between the log-layer and the edge of the boundary layer: a noticeable departure from the law of the wall occurs while approaching the freestream.

Viscous sublayer

The limiting behaviour of a turbulent flow in immediate vicinity of a wall (viscous sublayer) is considered. If y is the wall normal coordinate, the Taylor expansions of velocity and pressure are

$$
\begin{aligned}
u &= & a_1 y + a_2 y^2 + a_3 y^3 + ... \\
v &= & b_2 y^2 + b_3 y^3 + ... \\
w &= & c_1 y + c_2 y^2 + c_3 y^3 + ... \\
p &= & p_0 + p_1 y + p_2 y^2 + p_3 y^3 + ...
\end{aligned}
\tag{2.29}
$$

The velocity components in (2.29) satisfy the no-slip boundary conditions at the wall. Applying the Navier-Stokes equations, using (2.29), produces relations between the coefficients a_n etc.. From (2.29), it can be seen that in very-near-wall-region, where $a_1 y >> a_2 y^2 >> a_3 y^3 >> ...$, the relation between u and y is linear (viscous sublayer). The expansion (2.29) also holds for the fluctuating parts of a Reynolds-decomposition, which means the asymptotic behaviour of the Reynolds-stresses is:

$$
\begin{aligned}
\overline{u_1' u_1'} &= \overline{a_1' a_1'} y^2 + ... \\
\overline{u_2' u_2'} &= \overline{b_1' b_1'} y^4 + ... \\
\overline{u_3' u_3'} &= \overline{c_1' c_1'} y^2 + ... \\
\overline{u_1' u_2'} &= \overline{a_1' b_1'} y^3 + ...
\end{aligned}
\tag{2.30}
$$

Resulting in the following asymptotic behaviour for kinetic energy and dissipation rate:

$$
\begin{aligned}
k &= \frac{1}{2}\left((\overline{a_1' a_1'} + \overline{c_1' c_1'}) y^2 + 2(\overline{a_1' a_2'} + \overline{c_1' c_2'}) y^3 + ... \right) \\
\frac{\epsilon}{\nu} &= 2\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} \\
&= (\overline{a_1' a_1'} + \overline{c_1' c_1'}) + 4(\overline{a_1' a_2'} + \overline{c_1' c_2'}) y + ...
\end{aligned}
\tag{2.31}
$$

Two important conclusions result from equations (2.31). Firstly, it can be seen that the wall-value ($y = 0$) of the dissipation is non-zero, but more important is the remark

that both expressions in (2.31) can be combined to give the limiting value for ϵ:

$$\epsilon = 2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2 \qquad \text{for } y \to 0 \tag{2.32}$$

Log-layer

The law of the wall, which determines the velocity-profile in the log-layer, can be derived as described in the following. Consider a constant-pressure boundary layer (incompressible), where the relevant Navier-Stokes equations reduce to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} + \tau_{xy}^R \right)$$

In the log-layer, the convective terms are negligable, which means the sum of the viscous and Reynolds shear stress must be constant. Hence

$$\mu \frac{\partial u}{\partial y} + \tau_{xy}^R \approx \mu \left(\frac{\partial u}{\partial y} \right)_w = \tau_w = \rho u_\tau^2 \tag{2.33}$$

where w denotes the wall-value.

In the log-layer, the velocity profile should be determined by the wall situation, which means the velocity profile can be written as

$$\frac{u}{u_\tau} = f(\frac{y u_\tau}{\nu})$$

Derivation of this equation results in

$$\frac{du}{dy} = u_\tau \frac{df}{dy+} \frac{u_\tau}{\nu} \tag{2.34}$$

Consider equation (2.34) in the log-layer. In this region, the molecular viscosity should have no influence on the velocity-profile, if a turbulent viscosity is introduced which is independent of the molecular viscosity, because the Reynolds-stress is much larger than the viscous stress. This means that $\frac{\partial u}{\partial y}$ should be independent of ν. The only possible way of satisfying this constraint is by using a function f which satisfies

$$\frac{df}{dy+} = \frac{1}{\kappa y+} = \frac{\nu}{\kappa y u_\tau}$$

Substitution of this relation into (2.34) results in

$$\frac{du}{dy} = \frac{u_\tau}{\kappa y}$$

which can be integrated to what is called the law of the wall:

$$u^+ \approx \frac{1}{\kappa} ln y^+ + B \tag{2.35}$$

where the most common values for the contants are ([33]): $\kappa = 0.41$ and $B = 5.0$. κ is known as the Von Karman constant. The reason for this name is that when using the mixing-length model $\nu_t = l_M^2 \frac{\partial u}{\partial y}$, $\tau_{xy}^R = \nu_t \frac{\partial u}{\partial y}$, relation (2.33) reduces to

$$l_M^2 \left(\frac{\partial u}{\partial y} \right)^2 \approx u_\tau^2 \tag{2.36}$$

If the mixing length is given by the Von Karman relation $l_M = \kappa y$, equation (2.36) can be integrated to yield (2.35).

2.3.2   High-Reynolds modelling: boundary conditions in the log-layer

The standard k-ϵ model is only valid in regions where the turbulent Reynolds number R_t is sufficiently high ($R_t > 150$). This means that when moving away from the wall, these equations can only be applied starting from the log-layer. Practically, this means that in such calculations, the first grid-point does not lie on the wall as usual, but this first point has to be located in the log-layer, $30 < y_1^+ < 100$. In order to be able to perform computations, boundary conditions are necessary in this first grid point for velocity, k and ϵ. As the first grid point is assumed to be located in the log-layer, the law of the wall can be used to prescribe the velocity in the first grid point:

$$u(y_1) = u_\tau \left(\frac{1}{\kappa} ln y_1^+ + B \right) \tag{2.37}$$

In order to determine the boundary conditions for the turbulent quantities, production is assumed to be equal to dissipation, i.e. assuming that transport of turbulent kinetic

energy is negligible, and that the turbulent shear stress equals the wall stress:

$$\rho\epsilon \approx P_k \ , \qquad - \overline{u'v'} \approx \rho u_\tau^2$$

These assumptions can be translated into:

$$\rho\epsilon \approx -\overline{u'v'}\frac{\partial u}{\partial y} \approx \rho u_\tau^2 \frac{u_\tau}{\kappa y_1}$$

and, using the Boussinesq hypothesis:

$$u_\tau^2 \approx -\overline{u'v'} \approx c_\mu \frac{k^2}{\epsilon}\frac{\partial u}{\partial y} = c_\mu \frac{k^2}{u_\tau^2}$$

Thus, the boundary conditions for the turbulent quantities in the first grid-point are:

$$k(y_1) = \frac{u_\tau^2}{\sqrt{c_\mu}} \ \ , \qquad \epsilon(y_1) = \frac{u_\tau^3}{\kappa y_1} \tag{2.38}$$

2.3.3   Low-Reynolds k-ϵ modelling

In flows where the existence of universal wall functions is not established, for instance, turbulent boundary layer flows at low Reynolds numbers, unsteady flows and flows where separation occurs, low Reynolds-versions of the turbulence model equations should be formulated. It will also be illustrated in a later chapter that the choice of the position of the first grid point in high-Reynolds modelling can have a significant influence on the calculation results, which also illustrates the need for low-Reynolds modelling.

The aim of low-Reynolds modelling is being able to integrate the modelled k-ϵ equations up to the wall. In order to achieve this, modifications are introduced in the k-ϵ model in order to model the viscous interaction. The modifications usually consist of introducing damping functions, while trying to reproduce the correct asymptotic near-wall behaviour.

The general form for low-Reynolds k-ϵ turbulence models can be written as:

$$\begin{aligned}
\frac{\partial \rho k}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k k) &= \frac{\partial}{\partial x_k}\left(\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_k}\right) + P_k - \rho\epsilon - D \\
\frac{\partial \rho \epsilon}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k \epsilon) &= \frac{\partial}{\partial x_k}\left(\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_k}\right) + \frac{c_{\epsilon 1}f_1 P_k - c_{\epsilon 2}f_2\rho\epsilon}{\tau} + E_S
\end{aligned} \tag{2.39}$$

with

$$\mu_t = c_\mu f_\mu k\tau \qquad (2.40)$$

The example discussed here is the Yang and Shih model [2], which will be used further on. The main reason why this model was chosen, is the fact that it produces predictions of high quality in turbulent boundary layers (see for example [16]), mainly due to the fact that the tuning of the model was done using DNS-data instead of experimental data.

Considering the standard k-ϵ model is valid in the high-Reynolds region, the basic constants in (2.39) are the same as in the standard k-ϵ model: $c_\mu = 0.09$, $c_{\epsilon 1} = 1.44$, $c_{\epsilon 2} = 1.92$, $\sigma_k = 1.0$ and $\sigma_\epsilon = 1.3$.

The low-Reynolds modifications in the Yang-Shih model [2] consist of:

- The function f_μ

  Due to the presence of a wall, the turbulent length-scale is smaller in wall-region, which should be reflected in the turbulent viscosity. This is done by the introduction of a damping function f_μ which decreases from unity towards zero when approaching the wall. The asymptotic behaviour of f_μ in wall region should be such that the asymptotic limiting behaviour of the turbulent shear stress $(-\overline{u'v'} \sim \vartheta(y^3)$, see equation (2.30)) is satisfied. This is the case for the function proposed by Yang and Shih [2]:

  $$f_\mu = \sqrt{1 - exp(-1.5.10^{-4}R_y - 5.10^{-7}R_y^3 - 1.10^{-10}R_y^5)},$$
  $$R_y = \frac{\sqrt{k}y}{\nu}.$$

  where R_y is a parameter expressing the wall-distance. As $R_y \sim \vartheta(y^2)$, and thus $f_\mu \sim \vartheta(y)$, $\nu_t \sim \vartheta(y^3)$ (because $k \sim \vartheta(y^2)$ and $\tau \sim \vartheta(1)$), the correct limiting behaviour is obtained for $-\overline{u'v'} \sim \mu_t \frac{\partial u}{\partial y} \sim \vartheta(y^3)$.

- The function f_2

  This function expresses the influence of the Reynolds number on the decay-law $k \sim x^{-n}$. In the Yang-Shih model,

  $$f_2 = 1 - 0.22e^{(-(\frac{k\tau}{6\nu})^2)}$$

- The function f_1 and the term E_S

  The profile of the turbulent dissipation shows a local maximum in the buffer-layer, which causes a lower peak value of the turbulent kinetic energy in this region. This behaviour is mimicked by the use of f_1 and E_S. In the Yang-Shih model:

  $$f_1 = 1 \ , \quad E_S = \nu\nu_t(\frac{\partial^2 u_i}{\partial x_k \partial x_j})(\frac{\partial^2 u_i}{\partial x_k \partial x_j}) \ .$$

- The turbulent time-scale τ
  In the high-Reynolds model, this time-scale has the value $\tau_t = \frac{k}{\epsilon}$. In wall region, this time-scale has a zero limit, rendering a singular behaviour in the equation for ϵ. Considering the fact that both the length- and velocity-scales in wall-region are $\vartheta(y)$, the physical time-scale in wall-region has a non-zero value. Yang and Shih have assumed this value is the Kolmogorov time-scale $\tau_k = \sqrt{\frac{\nu}{\epsilon}}$ and have introduced a timescale

  $$\tau = \frac{k}{\epsilon} + \sqrt{\frac{\nu}{\epsilon}}$$

  which evoluates from the Kolmogorov time-scale in wall vicinity to the high-Reynolds turbulent time-scale far from the wall.

- The boundary conditions
  When low-Reynolds k-ϵ models are used, the equations are integrated directly up to the wall. This means boundary conditions need to be imposed on this wall. The boundary values of velocity and turbulent kinetic energy are of course zero (no-slip condition), while, in the Yang-Shih model, the boundary condition for the turbulent dissipation is taken in accordance with the wall limiting behaviour:

  $$u(0) = 0 , \quad k(0) = 0 ,$$
  $$\epsilon(0) = 2\mu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2$$

The elaborate discussion of the development of the standard k-ϵ constants follows in chapter 5.

2.4 Model development and validation using DNS

Although using DNS is at this time not feasible for general application, some DNS calculations have been performed for a variety of basic flows during the last decade (for example channel flow by Kim *et al.* [22]). These data are very valuable for turbulence modellers. Major advantages, compared to experimental data, are that all flow variables are accessible and uncertainties related to the influence of the experimental probe disappear. It is even possible to provide the budgets of Reynolds-stresses, turbulent kinetic energy and dissipation rate, as was done by Mansour *et al.* [76] for channel flow. These budgets can be a very valuable tool for modelling purposes.
DNS-data can not only be usefull for modelling purposes, but also for validation purposes. Recently, Le and Moin [30] have provided a DNS-database for a backward facing step flow at a low Reynolds number ($Re_h = 5100$). This forms an excellent test-case for testing the ability of turbulence models to predict this type of flow.