2. Nonlinear scalar problems

Consider the nonlinear PDE for the scalar function $u$:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$  \hspace{1cm} (1)

with initial data $\phi(x)$,

$$u(x,0) = \phi(x)$$  \hspace{1cm} (2)

In what follows we will assume that $f(u)$ is a convex function of $u$; i.e., we assume that

$$\frac{\partial^2 f}{\partial u^2} > 0$$

As an example we consider Burger’s equation for which

$$f(u) = \frac{1}{2} u^2$$

Let the wavespeed $a$ be defined by $a(u) = \frac{\partial f}{\partial u}$. Note that our convexity assume implies that $\frac{\partial a}{\partial u} > 0$. Assuming that $u(x,t)$ is differentiable, equation (1) can be rewritten in the quasilinear form:

$$\frac{\partial u}{\partial t} + a(u) \frac{\partial u}{\partial x} = 0$$  \hspace{1cm} (3)

Now consider solutions $x(t)$ to the ODE

$$\frac{\partial x}{\partial t} = a(u(x,t),t)$$  \hspace{1cm} (4)

$$\frac{\partial x}{\partial t} = a(u(x,t),t)\frac{\partial u}{\partial x}$$  \hspace{1cm} (5)

By differentiating $u(x(t),t)$ with respect to $t$ one can see that $u(x(t),t) = u(x-at,0) = \phi(x_0 - at)$ is a solution of (3). The function $x(t)$ is called a characteristic and the ODE (4)-(5) is called a characteristic equation. The characteristic can be thought of as either a trajectory in space or as a curve in space/time along which information (the solution $u$) propagates. Let’s examine what happens to $u$ along characteristics. Using the chain rule, we find that the material derivative of $u(x(t),t)$ is:

$$\frac{d}{dt} (u(x(t),t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial t}$$

$$\frac{d}{dt} (u(x(t),t)) = \frac{\partial u}{\partial t} + a(u(x,t),t) \frac{\partial u}{\partial x}$$  \hspace{1cm} (6)
However, the differential equation (3) tells us that the r.h.s. of equation (4) vanishes. Therefore, the derivative of $u$ is zero along a characteristic. This implies that $u$ is constant along characteristics.

![Figure 23. A characteristic curve in space-time.](image)

Now consider how the wavespeed $a$ for Burgers equation (1) changes along a characteristic. We have

$$\frac{dx}{dt} = a(u(x,t), t) = a(u(x_0, 0))$$

$$\frac{dx}{dt} = \phi(x_0) \quad (7)$$

since $u(x,t) = u(x_0,0)$ on the characteristic. Thus, for Burgers equation, the characteristics in space/time are straight lines with slope $\phi(x_0)$, and the information which propagates along the characteristics is $u(x_0,0)$. The difference between the linear and nonlinear cases is that the slopes of the characteristics are not necessarily constant for the latter case.

For example, it is apparent from Figure 24 that if the lines are extended further in time then eventually they will cross. Thus, if at time $t = 0$, there are some values of $u(x,0) = \phi(x)$ are positive and some are negative, or more generally if $\phi(x_1) > \phi(x_2)$ for some $x_1 < x_2$, then they will cross at some finite time. In general, whenever the initial data $\phi(x)$ is smooth and has compact support (i.e., $\phi(x) = 0$ outside of some finite interval), then there will be some critical time $T_c$ at which the solution $u$ is no longer single valued. At $T_c$ the value of $a(\phi)$ equivalent to the value carried with the left-running characteristic, the right-running characteristic, or both?
To examine this more closely, consider the behavior of $u$ in space. We know that the exact solution at any time $t$ is given by:

$$u(x, t) = \phi(x - a(u(x), t)t)$$

The partial derivative of $u(x(t), t)$ with respect to $x$ is:
where the prime indicates differentiation with respect to \( x \). Next use the chain rule to obtain

\[
\frac{\partial u}{\partial x} = \phi' - t \frac{d a}{d x} \frac{\partial u}{\partial x} \phi'
\]

or

\[
\frac{\partial u}{\partial x} = \frac{\phi'}{1 + t \frac{d a}{d u} \phi'}
\]  

(9)

If \( \phi' < 0 \) anywhere, then the spacial derivative of \( u \) blows up, since the denominator in equation (8) goes to zero at time \( t = T_c \) where

\[
T_c = -\frac{1}{\frac{d a}{d u} \phi'}
\]  

(10)

This analysis might lead on to believe that the solution cannot be continued beyond the time \( t = T_c \). However, from a physical standpoint, we know that there exist solutions to nonlinear hyperbolic equations beyond the critical time. In order to understand how one can extend the solutions to times \( t > T_c \), we need to introduce the concept of weak solutions.

### 2.1 Weak solutions of nonlinear hyperbolic problems

A weak solution to the nonlinear scalar equation (1) satisfies

\[
\int_{x_L}^{x_R} u(x, t^N) \, dx = \int_{x_L}^{x_R} u(x, t^0) \, dx + \int_{t^0}^{t^N} f(u(x, t)) \, dt - \int_{t^0}^{t^N} f(u(x, t)) \, dt
\]

(11)

between initial time \( t^0 \) and a later time \( t^N \) and between the left and right boundaries of the physical domain, \( x_L \) and \( x_R \) respectively. Consider the particular case of a discontinuous solution, specifically, an isolated discontinuity propagating at speed \( s \). From Figure 4, we can evaluate the integrals:

\[
\int_{x_L}^{x_R} (u(x, t^N) - u(x, t^0)) \, dx = u_L (\lambda_L - s (t^N - t^0)) + u_R (\lambda_R - s (t^N - t^0)) - (u_L \lambda_L + u_R \lambda_R)
\]

\[
\int_{x_L}^{x_R} (u(x, t^N) - u(x, t^0)) \, dx = (u_L - u_R) s (t^N - t^0)
\]  

(12)
and similarly
\[
\int_{t^0}^{t_N} (f(u(x_L, t)) - f(u(x_R, t))) \, dt = (f(u_L) - f(u_R)) \, (t^N - t^0)
\]
(13)

Using equations (11) and (12) above, we can solve for the propagation speed which the discontinuity is traveling at,
\[
s = \frac{f(u_L) - f(u_R)}{u_L - u_R}
\]
(14)

This is the known as the Rankine-Hugoniot jump relation. If \( f \) is a smooth function, then by the mean value theorem
\[
\frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{\partial f}{\partial u} \bigg|_{\xi}
\]
(15)

where \( \xi \) is some intermediate value between \( u_L \) and \( u_R \). Thus we see that as \( u_L \to u_R \), then
\[
s \to \frac{df}{du} \bigg|_{\frac{u_L + u_R}{2}} = a \left( \frac{u_L + u_R}{2} \right)
\]
(16)

This indicates that weak waves are propagated along characteristics.
2.1.1 **Nonuniqueness of weak solutions**

One problem with weak solutions is that they are not unique, i.e., for the same initial data, we can obtain more than one weak solution. For example, consider Burgers equation for which \( f(u) = \frac{1}{2} u^2 \) with the following initial data:

\[
u(x, 0) = \begin{cases} 
-1 & (x < 0) \\
1 & (x > 0)
\end{cases}
\]

The trivial solution is \( s = 0 \), shown in figure 5a. Another weak solution in which characteristics fan out from the point \( x = 0, t = 0 \) is presented in Figure 27b:

![Figure 27. Two weak solutions of Burgers equation: (a) a stationary shock; (b) a rarefaction fan.](image)

2.1.2 **The entropy condition**

We are thus left with the dilemma of deciding how to choose the physically correct solution. The answer to this conundrum is to require that the ‘correct’ weak solution be the limit of solutions to the associated viscous equation in the limit as the viscosity tends to zero. In other words, let \( u^\varepsilon \) be a solution of

\[
\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial f(u^\varepsilon)}{\partial x} = \varepsilon \frac{\partial^2 u^\varepsilon}{\partial x^2}
\]  

(17)

where \( \varepsilon \) is the viscosity coefficient. We require that \( u \) satisfy

\[
\lim_{\varepsilon \to 0} u^\varepsilon = u
\]

(18)
We claim that a weak solution which satisfies (17) leads to a well-posed problem. In other words, the solution exists, is unique, and has continuous dependence on initial data. Solutions which satisfy (17) are said to satisfy the entropy condition. This terminology comes from the fact that entropy can only increase across a gas dynamic shock and that (17) is one mechanism we can use to identify the solution to which has this property. These solutions are often referred to as ‘entropy solutions’.

\[ s = \frac{f(u_L) - f(u_R)}{u_L - u_R} \]  

where \( L \) and \( R \) denote left and right sides of the wave. For convex flux functions \( f; i.e., \frac{\partial^2 f}{\partial u^2} > 0 \), the entropy condition is satisfied if:

\[ a(u_L) > s > a(u_R) \]  

Figure 28. Weak solutions to Burgers equation which satisfy the entropy condition: (a) Converging characteristics; and (b) Rarefaction fan; and a weak solution which violates the entropy condition: (c) Diverging characteristics
(For Burgers equation, recall that \( a(u) = u \).) Geometrically, this means that the information must propagate forward in time as shown in Figure 26.

### 2.2 Strategies to enforce the entropy condition

To, in order to develop an adequate numerical scheme to solve the nonlinear scalar equation (1), we require that the numerical solution:

(i) be a weak solution of the problem;

(ii) be reasonably accurate in smooth regions using the design criteria for linear advection;

(iii) be stable, where we place more stringent requirements on the stability of the scheme for the case of nonlinear discontinuities (e.g., LW4 is stable for purely linear problems, but it spreads out the Fourier components of the wave since they travel at different speeds. This may cause problems such as mode-mode coupling in nonlinear problems);

(iv) satisfy the entropy condition.

Let’s examine what happens when the entropy condition is not satisfied. Consider the simple case of upwind differencing. The discretized form of the governing equation in conservation form is:

\[
\begin{align*}
  u_j^{n+1} &= u_j^n + \frac{\Delta t}{\Delta x} \left( F_{j-\frac{1}{2}}^n - F_{j+\frac{1}{2}}^n \right) \\
  F_{j+\frac{1}{2}} &= \begin{cases} 
  f(u_j) \ldots \text{if} \ldots a \left( \frac{u_j + u_{j+1}}{2} \right) > 0 \\
  f(u_{j+1}) \ldots \text{if} \ldots a \left( \frac{u_j + u_{j+1}}{2} \right) \leq 0
  \end{cases}
\end{align*}
\]

Desirable properties of this scheme include the facts that it is stable and total-variation-diminishing; however, it does not satisfy the entropy condition. To see this suppose we have the initial data at \( t = 0 \):

\[
  u_j^0 = \begin{cases} 
  -1 & j < 0 \\
  1 & j \geq 0
  \end{cases}
\]

Since this particular discontinuity violates the entropy condition, we would like the numerical simulation to dissipate the discontinuity with time. For Burgers equation, \( f(u_j) = u_j^2/2 \) and \( a(u_j) = \partial f/\partial u = u_j \) and a simple calculation shows that all the
fluxes cancel each other, and the solution reproduces itself at each time step. As $\Delta x$ and $\Delta t$ go to zero, we would converge to an entropy-violating shock using upwind differencing.

### 2.2.1 Artificial viscosity

There are a number of ways to cope with this problem. One solution is to add in a small amount of artificial viscosity, $\varepsilon$. The error in smooth regions is of order $\varepsilon$, while it is order $\sqrt{\varepsilon}$ at discontinuities. In general, if one chooses $\varepsilon = O(\Delta x)$ everywhere this will work. However, this will negate all of the careful improvements we have made in designing a high resolution numerical method. Therefore, to preserve the low phase error and other desirable properties of the methods discussed in Chapter I, we would like $\varepsilon = O(\Delta x)$ at the discontinuities, and $\varepsilon = O(\Delta x^p)$ in smooth regions, where $p$ is the order of the scheme. The truncation error for upwinding is:

$$
\frac{\partial u_{\text{MOD}}}{\partial t} + \frac{\partial f(u_{\text{MOD}})}{\partial x} = a(u_{\text{MOD}}) \left( \frac{1 - \sigma}{2} \right) \Delta x \frac{\partial^2 u_{\text{MOD}}}{\partial x^2}
$$

### 2.2.2 The first-order Godunov method

Another strategy is to utilize Godunov’s method, which is a “real” geometric upwinding scheme. The solution process is broken down into three steps:

(i) Interpolate $u_j(x)$ from $u_j^n$. Choose a piecewise linear interpolation function so that $u_I(x) = u_j^n$ for $(j - \frac{1}{2}) \Delta x < x < (j + \frac{1}{2}) \Delta x$. (See Figure 2 in Chapter 1);

(ii) Solve the above equation exactly; i.e., solve the Riemann problem at each discontinuity, with the left and right states given by $u_j$ and $u_{j+1}$ respectively. This is always solvable because of our assumption that $\partial^2 f/\partial u^2 > 0$ and $da/du > 0$. We will obtain compression waves or expansion fans as shown by Figure 6a and b above;

(iii) Average the results back on to the grid.

Step (ii) above ensures that this procedure produces a solution which satisfies the entropy condition, while step (iii) adds artificial viscosity. We will later prove that if we start out with something which satisfies the entropy condition and average it, the result will still satisfy the entropy condition. Each piece of the solution satisfies the entropy condition by itself and is a weak solution to the governing equation.

We now describe this procedure in more detail. We wish to solve the nonlinear scalar problem:

$$
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0
$$
The first-order Godunov finite-difference method for (24) is given by:

\[ u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} \left( F_{j-\frac{1}{2}} - F_{j+\frac{1}{2}} \right) \]  

(25)

where the flux is defined by

\[ F_{j+\frac{1}{2}} = f\left( \hat{u}_{j+\frac{1}{2}} \right) \]  

(26)

and \( \hat{u}_{j+\frac{1}{2}} \) is the solution to the Riemann problem; i.e., the solution of (24) with initial data

\[ u_j \]  

(27)

evaluated along the ray \( x/t = 0 \). This satisfies the entropy condition since by definition the exact solution of the Riemann problem is the entropy solution.

Let’s compare this method to classical upwinding. The flux for the latter is:

\[
F_{\text{upwind}}^{j+\frac{1}{2}} = \begin{cases} 
  f(u_j) & \text{if } a(u_j) > 0 > a(u_{j+1}) \\
  f(u_{j+1}) & \text{if } a(u_j) < 0 < a(u_{j+1}) 
\end{cases} 
\]

(28)

For cases in which the entropy condition is not violated, the result reduces to the same answer as upwinding would give. If \( a(u_j) > 0 > a(u_{j+1}) \), the Riemann problem would choose

\[ F_{\text{Godunov}}^{j+\frac{1}{2}} = \begin{cases} 
  f(u_j) & \text{if } s > 0 \\
  f(u_{j+1}) & \text{if } s < 0 
\end{cases} \]  

(29)

By the Rankine-Hugoniot relation for Burgers equation, \( f(u_j) = u_j^2/2 \), the speed is simply:

\[ s = \frac{u_j + u_{j+1}}{2} \]  

(30)

as for upwinding. On the other hand, if \( a(u_j) < 0 < a(u_{j+1}) \), solving the Riemann problem looks something like adding a viscous flux at the transonic point, so that we can write

\[ F_{\text{Godunov}}^{j+\frac{1}{2}} = F_{\text{upwind}}^{j+\frac{1}{2}} + \alpha \left( u_{j+1} - u_j \right) \]  

(31)

where \( \alpha > 0 \) and is of order \( |u_j| \) and \( |u_{j+1}| \). This adds a flux in the -x direction which tends to smooth out rarefaction shocks. Godunov’s method modifies the direction of informa-
tion transfer, depending on the sign of the wavespeed, just as upwinding does. However, Godunov’s method also tends to smooth out physically unrealistic rarefaction shocks.

2.2.3 The second-order Godunov method

The second-order Godunov’s method is a two-sided predictor/corrector scheme. This allows information to travel from both sides of the cell. First we must develop the concept of the approximate Riemann problem.

For convex functions, consider the case where the wavespeed in adjacent cells is increasing as \( a(u_j) < 0 < a(u_{j+1}) \). We would like to obtain \( \hat{u}_{j+\frac{1}{2}} \), which we define as the exact solution to \( a(\hat{u}_{j+\frac{1}{2}}) = 0 \). We can approximate the solution by linear interpolation:

\[
\hat{u}_{j+\frac{1}{2}} \approx u_j^n - \frac{a(u_j^n)}{a(u_{j+1}^n) - a(u_j^n)} (u_j^n - u_{j+1}^n)
\]  

(32)

For this case, there will be only one value of \( u \) for which \( a(u) = 0 \), as shown in figure 1, below.

Then the flux can be written

\[
f\left(\hat{u}_{j+\frac{1}{2}}\right) = f_{\text{upwind}}^{\alpha} + \alpha (u_j - u_{j+1})
\]  

(33)

where \( \alpha > 0 \), and is of order \( u_j \) and \( u_{j+1} \). To make this scheme second-order, we follow the same guidelines as for linear equations. Given \( u_j^n \) at time \( t^n \), we must find the interpolation

1. In other words, \( f'' > 0 \). This condition is met for the case of gasdynamics, but almost everything else in the real world is not convex. Examples of nonconvex functions are solutions to flow through porous media, magnetohydrodynamics, shocks in solids and phase change.
function; solve the initial-value problem for \( u_I(x) \) exactly; and average the results back down to the grid.

2.2.3.1 Outline of the method

Let's handle each of these steps sequentially. Assume that we know \( u_j^n \) at the cell centers, and that \( \Delta x \) and \( \Delta t \) are constants. We would like to evolve \( u \) such that

\[
u_j^{n+1} = \mathcal{L}(u^n)
\]

where \( \mathcal{L} \) is a nonlinear operator.

(i) Find the interpolation function \( u_I(x) \). The constraint on \( u_I(x) \) is that

\[
u_j^n = \frac{1}{\Delta x} \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} u_I(x) \, dx
\]

To find \( u_I(x) \), calculate

\[
u_I(x) = u_j^n + \frac{(x-j\Delta x)}{\Delta x} \Delta^{VL}_{u_j}
\]

where \( \Delta^{VL} \) is the van Leer slope. To construct the van Leer slopes, recall the definition from the discussion of geometric limiters for the linear scalar problem:

\[
\Delta^{VL}_{u_j} = \left\{ \begin{array}{ll}
S_j \cdot \min (2|u_{j+1} - u_j|, 2|u_j - u_{j-1}|, \frac{1}{2} |u_{j+1} - u_{j-1}|) & \text{if } (\phi > 0) \\
0 & \text{otherwise}
\end{array} \right.
\]

where

\[
S_j = \text{sign} (u_{j+1} - u_{j-1})
\]

and

\[
\phi = (u_{j+1} - u_j) \cdot (u_j - u_{j-1})
\]

The usage of flux limiters makes the scheme second order and, furthermore, helps us to avoid the nuisance of Gibb’s phenomenon at discontinuities.

(ii) Solve the initial-value problem for \( u_I(x) \) exactly. To accomplish this, calculate the left and right states for the Riemann problem, depending on the sign of the wavespeed:
These formulas were derived by considering upstream-centered Taylor series expansions on both sides. Consider a Taylor’s expansion for $u_{j+1/2,L}$.

$$u_{j+1/2,L} = \begin{cases} 
    u_j^n + \frac{1}{2} \left( 1 - a(u_j^n) \frac{\Delta t}{\Delta x} \right) \Delta V_L u_j & \text{if } (a(u_j) > 0) \\
    u_j^n + \frac{1}{2} \Delta V_L u_j & \text{if } (a(u_j) < 0)
\end{cases} \quad (39)$$

$$u_{j+1/2,R} = \begin{cases} 
    u_{j+1}^n - \frac{1}{2} \left( 1 + a(u_{j+1}^n) \frac{\Delta t}{\Delta x} \right) \Delta V_L u_{j+1} & \text{if } (a(u_{j+1}) < 0) \\
    u_{j+1}^n - \frac{1}{2} \Delta V_L u_{j+1} & \text{if } (a(u_{j+1}) > 0)
\end{cases} \quad (40)$$

Transform the temporal derivative to a spatial one by using the PDE in nonconservation form (i.e., equation (3)):

$$u_{j+1/2,L} = u_j^n + \frac{\Delta x}{2} \frac{\partial u_j}{\partial x} \bigg|_{j+1/2} + \frac{\Delta t}{2} \frac{\partial u_j}{\partial t} \bigg|_{j+1/2} + \ldots$$

where $a(u_j^n) > 0$. Finally we replace the spacial derivative of $u$ with a slope limited derivative,

$$u_{j+1/2,L} = u_j^n + \frac{1}{2} \left( 1 - a(u_j^n) \frac{\Delta t}{\Delta x} \right) \Delta V_L u_j \quad (41)$$

Had we been traveling to the left, $a(u_j^n) < 0$, and

$$u_{j+1/2,R} = u_{j+1}^n - \frac{1}{2} \left( 1 + a(u_{j+1}^n) \frac{\Delta t}{\Delta x} \right) \Delta V_L u_{j+1} \quad (42)$$

After having calculated the left and right states, solve the Riemann problem, i.e., find the half-step value $u_{j+1/2}^{n+1/2}$ along $x/t = 0$.

(iii) **Average the results back down to the grid.** To find the equivalent flux form, compute $u_{j+1/2}$ for $t^n < t < t^n + \Delta t$ and find

$$F_{j+\frac{1}{2}} = \frac{1}{\Delta t} \int_{t^n}^{t^n+\Delta t} f\left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) dt \quad (43)$$

Finally, calculate the updated value of the solution by conservative differencing.
\[ u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} \left( F_{j-\frac{1}{2}} - F_{j+\frac{1}{2}} \right) \]  

(44)

2.2.3.2 Analysis of the method

Let’s examine how this method performs in the following general categories of solution regimes:

(i) If all of the wave speeds are moving in one direction, will this scheme do the right thing? Will the wavespeeds \( a(u_j^n) \)'s have unambiguous sign?

(ii) What happens at shocks?

(iii) What happens at transonic expansions? Can this scheme generate an entropy-violating discontinuity?

One can see that the upwind state is always chosen by this method. When the wavespeed is positive \( a(u_j^n) > 0 \), information propagates from the left and conversely, when \( a(u_j^n) < 0 \) information propagates from the right.

\[ a > 0 \]

\[ a < 0 \]

(a) Shock wave  

(b) Expansion fan

Figure 30 Geometrical interpretation of Godunov’s method at discontinuities
This method is well behaved at shocks. Imagine the situation shown in Figure 8 above. The wavespeed $a$ changes sign across the shock. To deal with this, we extrapolate to the cell edge from both sides using a Taylor series expansion. This yields two different states at the cell edge. Next, treating these two extrapolated values as the left and right states of the Riemann problem, let the Riemann solver sort out the appropriate upwind value.

Now let’s examine the accuracy issue. We begin by examining the local truncation error in regions where the solution is smooth. Assume that the exact solution at the cell center is given by $u_{E,j}^n = u_e(j \Delta x, n \Delta t)$ and that $u_e$ satisfies the PDE

$$\frac{\partial u_E}{\partial t} + a(u_E) \frac{\partial u_E}{\partial x} = 0 \tag{45}$$

Further assume that $u_e$ is smooth enough (say it has two or three derivatives) and that the wave speed $a(u_e) > 0$. So far, we will make no restrictions on the convexity of $u_e$. We claim that, for second-order Godunov:

$$u_{E,j}^{n+1} - \mathcal{L} u_e^n = O(\Delta x^3) \tag{46}$$

Define the predictor step by

$$\tilde{u}_{E,j}^{n+\frac{1}{2}} = u_{E,j}^n + \frac{1}{2} \left( 1 - a(u_{E,j}^n) \frac{\Delta t}{\Delta x} \right) \Delta u_{E,j}^n \tag{47}$$

and define

$$u_{E,j}^{n+\frac{1}{2}} = u_e \left( (j + \frac{1}{2}) \Delta x, (n + \frac{1}{2}) \Delta t \right) \tag{48}$$

or, rewriting as a Taylor series expansion about $j \Delta x, n \Delta t$:

$$u_{E,j}^{n+\frac{1}{2}} = u_{E,j}^n + \frac{\Delta x}{2} \frac{\partial u_E}{\partial x} \bigg|_{j \Delta x, n \Delta t} + \frac{\Delta x^2}{8} \frac{\partial^2 u_E}{\partial x^2} \bigg|_{j \Delta x, n \Delta t} + \frac{\Delta t}{2} \frac{\partial u_E}{\partial t} \bigg|_{j \Delta x, n \Delta t} + \frac{\Delta t^2}{8} \frac{\partial^2 u_E}{\partial t^2} \bigg|_{j \Delta x, n \Delta t} + \frac{\Delta x \Delta t}{4} \frac{\partial^2 u_E}{\partial x \partial t} \bigg|_{j \Delta x, n \Delta t} + O(\Delta x^3, \Delta t^3) \tag{49}$$

Since $u_e$ is a smooth function and satisfies the PDE:

$$\frac{\partial u_E}{\partial t} \bigg|_{j \Delta x, n \Delta t} = -a(u_E) \frac{\partial u_E}{\partial x} \bigg|_{j \Delta x, n \Delta t} \tag{50}$$

Then, substituting equation (50) into (49) and regrouping, we obtain,
We know that as $\Delta x \to 0$ and $\Delta t \to 0$, the ratio of $\Delta t$ to $\Delta x$ goes to a constant value so that $O(\Delta t^3) \to O(\Delta x^3)$. Comparing the expressions for (51) and twiddle (47) above, and noting that

$$\Delta u_E^n_j - (\Delta x) \frac{\partial u_E}{\partial x} \bigg|_{j\Delta x, n\Delta t} = O(\Delta x^3)$$

if the derivative is approximated with central differencing, we conclude that

$$u_{E,j}^{n+\frac{1}{2}} - u_{E,j}^{n-\frac{1}{2}} = C \left[ (j + \frac{1}{2}) \Delta x \right] O(\Delta x^3)$$

where $C$ indicates some smooth function. Notice that the term in brackets is of order $\Delta x^3$.

Now we need to show that the local truncation error of the evolved solution is of order $\Delta x^3$. Using central differencing, we know that:

$$u_{E,j}^{n+1} - u_{E,j}^n = \frac{\partial u_E}{\partial t} \bigg|_{(j + 1/2) \Delta x, (n + 1/2) \Delta t} + O(\Delta t^2)$$

and

$$f(u_{E,j+1/2}^{n+1/2}) - f(u_{E,j-1/2}^{n+1/2}) = \frac{\partial f}{\partial x} \bigg|_{(j + 1/2) \Delta x, (n + 1/2) \Delta t} + O(\Delta x^2)$$

Then the evolved solution is

$$L(u_{E,j}^n) = u_{E,j}^n + \frac{\Delta t}{\Delta x} [f(\tilde{u}_{j-1/2}) - f(\tilde{u}_{j+1/2})]$$

$$L(u_{E,j}^n) = u_{E,j}^n + \frac{\Delta t}{\Delta x} [f(u_{E,j+1/2}^{n+1/2}) - f(u_{E,j-1/2}^{n+1/2})] + C_1 \left[ (j + \frac{1}{2}) \Delta x \right] + C_2 \left[ (j - \frac{1}{2}) \Delta x \right] + O(\Delta x^3)$$

2.2.4 The convexification of the Riemann problem

The next question to ask is what happens if $f'$ changes sign? (If $f'' < 0$, the condition for convexity is not met.) The answer is that weak solutions still exist, but entropy conditions become much harder to enforce. The Lax entropy condition ($a(u_j) \geq s \geq a(u_{j+1})$) is a necessary but not sufficient condition.
Let’s examine the Riemann problem with some \( f(u) \) which is monotonically increasing but has a change in curvature. In pictures:

![Figure 31: Pictorial representation of \( f(u) \)](image)

The goal is to construct a new function, \( \tilde{f}(u) \) between \( u_L \) and \( u_R \) which is the smallest convex function which is greater than or equal to \( f \). Then solve the Riemann problem with the Lax entropy condition on the new convex function \( \tilde{f}(u) \).

This function \( \tilde{f}(u) \) turns out to be either equal to \( f \) or is linear. For example

![Figure 32: Convexification of the function \( f(u) \)](image)
Next, place discontinuities at the intersections between the regions where \( \tilde{f}(u) \) follows \( f \) and where it is linear (locations \( s \) and \( s+1 \) in Figure 10). The wavespeed of the discontinuity is

\[
\frac{d\tilde{f}}{du} = \frac{f(u_{s+1}) - f(u_s)}{u_{s+1} - u_s}
\]  

(57)

If \( u_L > u_R \), this will look like a shock; if the opposite holds true, then \( \tilde{f}(u) = f \) and behavior like an expansion fan will be seen. This however is a lot of work. We discuss an approach in the next section.

2.2.5 The Engquist-Osher flux

Consider the flux obtained for first-order Godunov:

\[
u_{j+1}^n = u_j^n + \frac{\Delta t}{\Delta x} \left[ f(u_{RP_{j-1/2}}) - f(u_{RP_{j+1/2}}) \right]
\]  

(58)

where \( u_{RP} \) is the solution for the Riemann problem with left and right states \( u_j^n \) and \( u_{j+1}^n \).

We would like to find a flux function that is:

(i) upwind when the sign of \( a \) is unambiguous;
(ii) always more dissipative than Godunov;
(iii) easier to deal with than Godunov.

One solution is to use the Engquist-Osher flux, \( f_{EO} \), where

\[
f_{EO} = f(u_L) + \int_{u_L}^{u_R} \min (a(u), 0) \, du
\]  

(59)

where \( a(u) = df/du \). The term \( f_G \) is the Godunov flux. This indicates that \( f_{EO} \) is simply the Godunov flux plus some diffusive function of \( (u_L-u_R) \). Let’s examine the specific cases which arise when \( f(u) \) is convex \( (f'' > 0) \):

(i) \( a(u) > 0 \): \( f_{EO} = f_G = f(u_L) \)
(ii) \( a(u) < 0 \): \( f_{EO} = f_G = f(u_R) \)
(iii) \( a(u_L) < 0 < a(u_R) \): \( f_{EO} = f_G = f(a^{-1}(0)) \) where \( a^{-1}(0) \) is the value corresponding to \( u=0 \)?
(iv) \( a(u_L) > 0 > a(u_R) \): This transonic case is the interesting one. For this case, \( f_G = f(u_L) \) or \( f(u_R) \), depending on the sign of \( s \). However \( f_{EO} \) is given by:

\[
f_{EO} = f(u_L) + \int_{u_L}^{u_R} \min (a(u), 0) \, du
\]
Note that for this case $f_{EO}$ and $f_G$ are different. For Burger’s equation, $f(u) = u^2/2$ and

\[ f_{EO} = f(u_L) + \int_{a^{-1}(0)}^{u_R} \min(a(u), 0) \, du \]

\[ f_{EO} = f(u_L) + \int_{a^{-1}(0)}^{u_R} \min(a(u), 0) \, du \]

\[ f_{EO} = f(u_L) + f(u_R) - f(a^{-1}(0)) \]  \hspace{1cm} (60)

Note that for this case $f_{EO} \neq f_G$. For Burger’s equation, $f(u) = u^2/2$ and

\[ f_G = \begin{cases} 
\frac{u_L^2}{2} & \text{if } \ldots (s > 0) \\
\frac{u_R^2}{2} & \text{if } \ldots (s < 0)
\end{cases} \]  \hspace{1cm} (61)

and

\[ f_{EO} = \frac{u_L^2}{2} + \frac{u_R^2}{2} \]  \hspace{1cm} (62)

Since $f_{EO}$ is larger than $f_G$, it is more diffusive and can push more “stuff” down gradients. This indicates that the Engquist-Osher flux will spread out discontinuities, but only for the case of transonic flow; however, the numerical diffusion will be somewhat muted by the characteristics which will drive the discontinuities back in. This is advantageous over the convexification approach described in the last section, since there is much less logic involved. In addition, the logic that is involved is simply algebra; once we find $a(0)$ for the first time, we can simply store it and we’re set. For further details, see Bell, Colella and Trangenstein (1989).

The drawback to this approach is the difficulty in extending the procedure to higher order. Osher has shown that the only known schemes which are known to produce entropy-satisfying shocks for a general $f$ are first order while Bell and Shubin have made the empirical observation that using van Leer flux limiting will lead to entropy-violating discontinuities. We know that for convex $f$, entropy violations are very unstable, however for nonconvex $f$, entropy violations can actually steepen the shock. The answer lies in changing the limiter. For example, one might try some kind of modified van Leer approach. If either $f' > 0$ or $f'' < 0$ over large intervals of $u$, one might be able to reduce the scheme to first order only near points where $f'' = 0$ and retain higher-order accuracy elsewhere.