Near-Wall Velocity (Wall Functions):

Assumptions:
1.) the flow is (quasi-)steady;
2.) there are no streamwise ($x$-direction) pressure gradients (or mild streamwise $\Delta p$) => no separation;
3.) the fluid velocity is directed parallel to a flat wall and varies only in the direction normal to the wall (i.e., fully developed flow, $\frac{\partial}{\partial x} = 0$);

Consider a 2D incompressible turbulent flow near the wall

The continuity equation yields:
\[
\frac{\partial V}{\partial y} = 0 \quad (1)
\]

The $x$-momentum equation yields:
\[
0 = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \left( v \frac{\partial U}{\partial y} - \bar{u} \bar{v}' \right) \quad (2)
\]

The $y$-momentum equation yields:
\[
0 = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{\partial}{\partial y} (-\bar{v} \bar{v}') \quad (3)
\]

From equation (1) => $V = f_y(x)$ & since $V = 0$ at $y = 0$ gives $f_y(x) = 0 \Rightarrow V = 0$.

From equation (3)
\[
\frac{\partial}{\partial y} \left( \frac{P}{\rho} + \bar{v}'\bar{v}' \right) = 0 \quad \Rightarrow \quad \frac{P}{\rho} + \bar{v}'\bar{v}' = f_p(x) \quad (4)
\]

From equation (4)
\[
\frac{\partial}{\partial x} \left[ \frac{P}{\rho} + \bar{v}'\bar{v}' = f_p(x) \right]
\]
\[
\frac{\partial}{\partial x} \left( \frac{P}{\rho} \right) + \frac{\partial}{\partial x} (\bar{v}'\bar{v}') = \frac{df_p}{dx}
\]  \quad (5)

Since \( \frac{\partial}{\partial x} (\bar{v}'\bar{v}') = 0 \), gives
\[
\frac{\partial}{\partial x} \left( \frac{P}{\rho} \right) = \frac{df_p}{dx} \quad (6)
\]

From equation (2) & (6), we have
\[
\bar{v}' \frac{\partial U}{\partial y} - \bar{u}'\bar{v}' = \left[ \frac{d}{dx} \left( \frac{P}{\rho} \right) \right] y + f_u(x) \quad (7)
\]

Since at \( y = 0 \), \( \bar{u}'\bar{v}' = 0 \), we have
\[
\left. \bar{v}' \frac{\partial U}{\partial y} \right|_{y=0} = f_u(x) \quad (8)
\]

By definition (wall shear stress, \( \tau_w \)),
\[
\left. \bar{v}' \frac{\partial U}{\partial y} \right|_{y=0} = \frac{\tau_w}{\rho} \quad (9)
\]

Define the (wall-)friction velocity \( (u_\tau) \) as
\[
u_\tau = \sqrt{\frac{\tau_w}{\rho}}
\]
and let

\[ U^+ = \frac{U}{u_\tau}, \quad y^+ = \frac{yu_\tau}{\nu}, \quad x^+ = \frac{xu_\tau}{\nu} \]

\[ P^+ = \frac{P}{\rho u_\tau^2}, \quad \overline{u'v'}^+ = \frac{\overline{u'v'}}{u_\tau^2} \]

then equation (7) becomes

\[ \frac{\partial U^+}{\partial y^+} - \overline{u'v'}^+ = \left( \frac{dP^+}{dx^+} \right) y^+ + 1 \]

(10)

Accordingly,

\[ U^+ = f \left( \overline{u'v'}^+, \frac{dP^+}{dx^+}, y^+ \right) \]

or

\[ U = f \left( \overline{u'v'}, \frac{dP}{dx}, y, \nu, \frac{\tau_w}{\rho}, \text{boundary conditions} \right) \]

(11)

\[ \star \] By the Boussinesq approximation

\[ -\overline{u'v'} \propto \frac{\partial U}{\partial y} \]

or

\[ -\overline{u'v'}^+ = \kappa y^+ \left( \frac{\partial U^+}{\partial y^+} \right) \]

(12)
where $\kappa$ is called the von Kàrmàn constant and is an empirical constant.

In the log layer, $y^+ < 10$

\[ v_t \approx v \quad \text{i.e.,} \quad -\overline{u'v'} = v\frac{\partial U}{\partial y} \quad (13) \]

(12) & (13) into (10) gives

\[ \frac{\partial U^+}{\partial y^+} = \frac{\left( \frac{dP^+}{dx^+} \right) y^+ + 1}{\kappa y^+} \quad (14) \]

For a mild pressure gradient near the wall, we have

\[ \frac{dP^+}{dx^+} y^+ = 1 \quad (15) \]

or for a zero pressure gradient near the wall

\[ \frac{dP^+}{dx^+} y^+ = 0 \quad (16) \]

Thus, equation (14) reduces to

\[ \frac{\partial U^+}{\partial y^+} = \frac{1}{\kappa y^+} \quad (17) \]

integrating, we have

\[ U^+ = \frac{1}{\kappa} \ln(y^+) + c \quad (18) \]
where the von Kàrmàn constant and the integration constant (c) are determined from experiments, e.g.,

(1) For boundary layer flow over smooth wall and \( y^+ > 10 \)

\[
U^+ = 2.44 \ln(y^+) + 4.9 \quad \kappa = 0.4 \sim 0.43 \quad (19)
\]

(2) For pipe flow with a pressure gradient and \( y^+ > 10 \)
(a) for smooth wall

\[
U^+ = 2.5 \ln(y^+) + 5.5 \quad (20)
\]

(b) for rough wall

\[
U^+ = 2.5 \ln(y^+) + 8.5 \quad (21)
\]

\( \star \) For \( y^+ < 10 \), i.e., very close to the wall (viscous sublayer)

\[
v_t \ll v \quad \text{i.e.,} \quad -u'v' \ll v \frac{\partial U}{\partial y}
\]
then the x-momentum equation (equation 2) reduces to

\[ 0 = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \left( \nu \frac{\partial U}{\partial y} \right) \]  
(22)

For mild or no pressure gradients near the wall

\[ 0 = \frac{\partial}{\partial y} \left( \nu \frac{\partial U}{\partial y} \right) \]  
(23)

integrating, we have

\[ \nu \frac{\partial U}{\partial y} = c_1 = \frac{\tau_w}{\rho} \]

\[ U = \frac{\tau_w y}{\rho \nu} + c_2 \quad \text{Since} \quad U(0) = 0 \quad \Rightarrow c_2 = 0 \]  
(24)

\[ U = \frac{\tau_w y}{\rho \nu} \]

Also \( u_\tau = \sqrt{\frac{\tau_w}{\rho}} \), (24) becomes

\[ \frac{U}{u_\tau} = \frac{u_\tau y}{\nu} \]  
(25)

recall

\[ + = \frac{U}{u_\tau} \quad y^+ = \frac{y u_\tau}{\nu} \]

gives

\[ U^+ = y^+ \]  
(26)
Summary:

\[ U^+ = y^+ \quad \text{if} \quad y^+ < 10 \]
\[ U^+ = \frac{1}{k} \ln(y^+) + c \quad \text{if} \quad y^+ \geq 10 \]  

Near-Wall $\overline{u'v'}$ and $k$

Near the wall, turbulent fluctuation is small, whereas production and dissipation are large. Therefore, the $k$ equation

\[
\text{unsteady + convection} = \frac{\partial U_i}{\partial x_j} - \rho \varepsilon + \text{diffusion}
\]

reduces to

\[
(-u'v') \frac{\partial U}{\partial y} - \varepsilon = 0
\]  

\[
(-u'v') \frac{\partial U}{\partial y} = \varepsilon
\]

i.e., equilibrium (production = dissipation). From the $x$-momentum equation:

\[
0 = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \left( \nu \frac{\partial U}{\partial y} - \overline{u'v'} \right)
\]  

For mild or no pressure gradient flow,

\[
\nu \frac{\partial U}{\partial y} - \overline{u'v'} = f_u(x)
\]  

From (8) and (9),

\[
\nu \frac{\partial U}{\partial y} - \overline{u'v'} = \frac{\tau_w}{\rho}
\]
In the log layer,

\[ v_t \gg v, \quad \text{i.e.,} \quad -u'v' = v \frac{\partial U}{\partial y} \]

gives,

\[ -u'v' = \frac{\tau_w}{\rho} = u_\tau \]  \hspace{1cm} (32)

Recall

\[ -u'v' = v_t \frac{\partial U}{\partial y} = \left( C_\mu \frac{k^2}{\epsilon} \right) \frac{\partial U}{\partial y} \]  \hspace{1cm} (33)

Combining (28), (32), and (33), yields

\[ -u'v' = \frac{(C_\mu k^2) \frac{\partial U}{\partial y}}{-u'v' \frac{\partial U}{\partial y}} \]  \hspace{1cm} (34)

or

\[ (-u'v')^2 = C_\mu k^2 \]  \hspace{1cm} (35)

Therefore, near the wall along with \( -u'v' = \frac{\tau_w}{\rho} = u_\tau^2 \),

\[ k = \frac{\tau_w}{\rho} \frac{1}{\sqrt{C_\mu}} = \frac{u_\tau^2}{\sqrt{C_\mu}} \]  \hspace{1cm} (36)

\[ \text{\textbullet \quad \text{Summary: For } y^+ > 10,} \]

\[ k = \frac{u_\tau^2}{\sqrt{C_\mu}} \quad \text{and} \quad -u'v' \Big|_{\text{wall}} = \frac{\tau_w}{\rho} = u_\tau^2 \]  \hspace{1cm} (37)
Near-Wall Velocity Gradient and $\varepsilon$:

From (28) and (13)

$$(-u'v') \frac{\partial U}{\partial y} = \varepsilon \quad (38)$$

$$-u'v^+ = \kappa y^+ \left( \frac{\partial U^+}{\partial y^+} \right) \quad \text{or} \quad -u'v = \kappa y u_\tau \left( \frac{\partial U}{\partial y} \right) \quad (39)$$

gives

$$\frac{(-u'v')^2}{\kappa y u_\tau} = \varepsilon \quad (40)$$

Using (32), we have

$$\varepsilon = \frac{(u_\tau^2)^2}{\kappa y u_\tau} = \frac{u_\tau^3}{\kappa y} \quad (41)$$

From (39) and using (32),

$$\frac{\partial U}{\partial y} = \frac{-u'v}{\kappa y u_\tau} = \frac{u_\tau^2}{\kappa y u_\tau} = \frac{u_\tau}{\kappa y} \quad (42)$$

Summary: For $y^+ > 10$, 

$$\varepsilon = \frac{u_\tau^3}{\kappa y} \quad \text{and} \quad \frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y} \quad (43)$$
Summary of the Wall Bdary Conditions:

- High-Re Model

The log-law-based relations are adapted to bridge the semi-viscous near-wall region; first point

\[ 10 \leq n^+ \leq 100 \quad (30 \leq n^+ \leq 300) \]

The wall-parallel velocity normalized by the wall-friction velocity:

\[ U^+ = \frac{1}{\kappa} \ln(n^+) + c \]

\[ U^+ = \frac{U}{u_\tau}, \quad n^+ = \frac{nu_\tau}{\nu} \quad (44) \]

Velocity Gradient:

\[ \frac{\partial U}{\partial n} = \frac{u_\tau}{\kappa n} \quad (45) \]

Reynolds-shear stress:

\[ -\overline{u'v'} \bigg|_{\text{wall}} = u_\tau^2 \quad (46) \]

Local energy equilibrium

\[ \varepsilon = \frac{u_\tau^3}{\kappa n} \quad (47) \]
In log-law region

\[ k = \frac{u_\tau^2}{\sqrt{C_\mu}} \quad \Rightarrow \quad u_\tau = C_\mu^4 k^{\frac{1}{2}} \quad (48) \]

- Low-Re Model

The integration is carried out up to the wall through the viscous sublayer

\[ \left( U_i \right)_{wall} = 0; \quad \left( \frac{\partial P}{\partial n} \right)_{wall} = 0 \]

\[ \left[ \frac{\partial (u_i' u_j')}{\partial n} \right]_{wall} = 0; \quad \left( \frac{\partial k}{\partial n} \right)_{wall} = 0 \]

\[ \varepsilon_{wall} = 2\nu \left( \frac{1}{2} \frac{\partial k^2}{\partial n} \right)_{wall} \]
Estimation of the 1st Grid Point $y^+$:

$$\frac{1}{f^{1/2}} = -2.0\log\left(\frac{\zeta/d}{3.7} + \frac{2.51}{Re_d f^{1/2}}\right)$$

$$\frac{1}{f^{1/2}} \approx -1.8\log\left[\left(\frac{\zeta/d}{3.7}\right)^{1.11} + \frac{6.9}{Re_d}\right]$$

where

$$f = \frac{\tau_w}{\frac{1}{2}\rho V}$$

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

$$y^+ = \frac{y u_\tau}{v}$$