Shearing flow over an idealized wavy surface: Comparison between linear theory and DNS.

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ABSTRACT

Laminar Couette flow over a fixed wavy surface was studied with direct numerical simulation (DNS) in a 2D numerical domain. The mesh was generated by a conformal transformation that sets horizontal flow at the top of the domain, where a velocity boundary condition is given. The bottom of the domain is a wavy surface of wave slope \( ak \) in the range \((0.01 \leq ak \leq 0.35)\) and Reynolds numbers \( Re=10^2 \) and \( 10^3 \). The velocity profiles, the normal pressure and the shear stress (over the wall) were calculated and compared with theory. The goal is to explore the limit of validity of the linear theory [2] as the control parameter \( ak \) increases. The results show how an increase in wave slope produces an increase in the normalized shear stress and dimensionless pressure over the surface of the wave. The limit of validity was found approximately at \( ak=0.2 \) for \( Re=10^3 \) and \( ak=0.35 \) for \( Re=10^5 \) where recirculation appears at the troughs. The linear approximation is very accurate for the dual limit of very large Reynolds numbers and small wave slopes, while its accuracy decreases with smaller Reynolds numbers.

INTRODUCTION

When two immiscible fluids interact through a common interface, as is the case of wind-wave interactions in the ocean or sand dunes in the sea bottom, it is important to understand the basic mechanisms responsible for the complex behavior observed in nature. In particular, the shearing stresses and the normal pressure acting on the interface are responsible for the geometry modification in a free (or mobile) surface creating waves.

Flow over wavy surfaces is a subject that has been widely investigated in the past, however most of the studies reported in the literature have limitations when nonlinear effects have to be considered. Miles [1] studied the generation of surface waves by shear flows solving a boundary value problem obtaining results that agree with the observations in a qualitative way. Benjamin [2] produced an accurate linear theory for calculating the normal and tangential stresses on the boundary of a simple-harmonic wavy surface produced by shearing flows for stable laminar flows (and for turbulent flows considered as ‘pseudo-laminar’ using the mean-velocity as velocity profile). The validity of the aforementioned theory is limited to large Reynolds numbers, small amplitudes and it is assumed that the thickness of the boundary layer is much smaller than the wavelength.

Similar studies have been made using channels with a wavy wall; Nakayama and Sakio [3] studied a pressure derived flow using two modes of two-dimensional cosine waves, with different amplitudes and wavelengths at the lower boundary. Malevich et al. [4] used mathematical analysis to study Couette flow in a similar channel, in order to investigate the Reynolds numbers and wave amplitudes on the onset of recirculation. This situation has also been treated by Zhou et al. [5] for Poiseuille flow (and referencesence therein), using a perturbation technique for small wave amplitudes and a finite element numerical code for large perturbations, sinusoidal, triangular and arched-shaped upper surfaces with a flat bottom.

From the numerical point of view, Cherukat et al. [6] also studied a turbulent flow over a solid train of waves using a spectral element DNS technique. Flow separation and a variety of flow patterns were observed to be in agreement with the observations reported in the literature. Sullivan et al. [7] has also used DNS for the study of a turbulent flow over idealized water waves represented by the lower wall of a Couette channel using different wave slopes. His results agree with existing experiments and other simulations and show that the mean flow, velocity variances, pressure and drag vertical momentum fluxes are significantly influenced by the wave geometry and phase velocity \( c \) (normalized with the wave slope \( ak \), and the wave age \( c/\nu^* \), where \( \nu^* \) is the friction velocity defined as \( U_* = \sqrt{\nu_*/\rho} \).

In this report we show some results related with the effect of a wavy surface on a flow, in the context of a larger project aimed to simulate sudden roughness changes and internal boundary layers in the atmosphere: this phenomenon is usually studied numerically using turbulence modelling (see Rao et al. [8], Moeng [9]), however in this case DNS is proposed as a tool capable of making all field variables available for analysis and comparison with experiments in a smaller scale (Chamorro and Porté-Angel [10]).

The present results are limited to assess the limits of some analytical approximations available in the literature [2], in order to understand the mechanisms that dominate the onset of recirculation zones in the trough, without a transition to the turbulent regime.

Here, the term DNS is used to describe 2D simulations, which for the case of a laminar flow at high Reynolds numbers does not present 3D features characteristic of turbulent flows. The 2D representation has proved to be
pertinent even for some turbulent flows where the mean flow has only one component (e.g. see Rao et al. [8]).

The numerical code used here (JADIM) was developed in the Institut de Mecanique des Fluides de Toulouse (IMFT) by a team group lead by Jacques Magnaudet and Dominique Legendre. The code can solve the 3D Navier-Stokes equations for incompressible and unsteady situations in terms of velocity-pressure variables. The discretization method is implemented with finite volumes, which is well adapted to properties conservation. Precision is second order in time and space (Runge–Kutta/Crank–Nicolson schemes) and the code has been used to solve hydrodynamic and transfer problems in the past (see Magnaudet et al., [11]; Calmet and Magnaudet, [12]; Legendre and Magnaudet, [13]; Legendre et al., [14]; Figueroa-Espinoza and Legendre, [15].

METODOLOGY

Boundary fitted grids were produced using a conformal transformation that maps to the geometry shown in Fig. 1, where the south boundary represents a wavy surface of amplitude $a$. The transformed variables were taken from the potential and stream functions of a progressive linearized wave that propagates without change of form in a steady flow [16] (up side down and shifted L/2 in X direction), which are given by Eq. (1) and (2):

\[
\psi(X + \frac{L}{2}, -Y) = cY - \frac{ag \sinh k(h-y)}{\cosh kh} \cos(kX),
\]

\[
\phi(X + \frac{L}{2}, -Y) = cX - \frac{ag \cosh k(h-y)}{\cosh kh} \sin(kX),
\]

were the phase velocity is represented by $c$, the angular frequency by $\sigma$, gravity by $g$, the sea depth by $h$ (channel height in our case) and the wave number by $k = 2\pi/L$. We use the Eq. (1)-(2) to define the geometry of the conformal mapping; the geometry is fixed (here $c$, $g$ and $\sigma$ are parameters that serve the purpose of grid generation).

The width of cells used in the mesh was constant in the direction of $\phi$ (horizontal) and variable in the direction of $\psi$ (vertical) with minimal values near the walls and maximum value in the center of the mesh. The rate of change of the width in the of $\psi$ direction is determined by $S = \frac{\Delta \phi_i}{\Delta \phi_{i+1}} = 1.03$, were $\Delta \phi_i$ and $\Delta \phi_{i+1}$ are the width of a given cell and her upper neighbor cell, respectively.

![Diagram of the mesh produced by a conformal transformation.](image)

The east and west boundary conditions are periodic, with a wavelength $\lambda$. The north boundary is flat and a constant velocity $U_m$ is imposed at height $h$. The south boundary condition is a no-slip condition and the initial condition (at time zero) is a linear (Couette) velocity profile. The flow was assumed incompressible and gravity was not considered.

Convergence tests were carried out for different grid parameters like the number of nodes in both vertical and horizontal directions. Grid independence was found for $n_x = n_y = 60$, and $Re=10^4$, where $n_x$ and $n_y$ are the number of nodes following the horizontal and vertical directions respectively and the Reynolds number is given by $Re = \rho U h/\mu$ were $\mu$ is the dynamic viscosity.

Simulations were run for different wave slopes and Reynolds numbers, most of which are listed in Table 1 (0.01 $\leq ak \leq 0.35$, $100 \leq Re \leq 10000$).

The JADIM code provides the values of pressure $p$, vertical and horizontal velocities, $u$ and $v$ respectively, for each node. The wall shear stress was calculated from eq. (3):

\[
\tau_w = \mu \left( \frac{\partial U}{\partial \phi} \right)_{\phi=0},
\]

where $U$ and $V$ are the covariant velocity profiles, given by eq. (4) and (5),

\[
U = \frac{u \partial \phi}{\partial x} + \frac{v \partial \phi}{\partial y},
\]

\[
V = \frac{u \partial \phi}{\partial x} + \frac{v \partial \phi}{\partial y},
\]

where $J$ is the transformation Jacobian:

\[
J = \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2.
\]

The transformed variables were used in Eqns. (3)-6 because they fit the position and orientation of the walls (the velocity profiles were fitted using a polynomial regressions of $e^{th}$ order for the velocity profile function of $U$ in the vicinity of the wall).

RESULTS

Figure 2 shows the dimensionless shear stress, while Fig. 3 shows the dimensionless pressure, both variables given as a function of dimensionless length $X\lambda$, and normalized by the mean friction velocity given by $U_* = \sqrt{\tau_{av}/\rho}$. Here $\tau_{av}$ is the average shear stress along the wall. Nonlinear effects grow progressively as the wave slope $ak$ increases with a fixed Reynolds number, until a change in flow configuration is observed when the shear stress near the trough shifts sign as shown in the lower part of Fig. 2. In Fig. 2 and 3 the linear theory of Benjamin (equations 5.6 and 5.9 in his paper [2]) are also shown.

In Fig. 2 and 3 it can be seen that linear theory represents an adequate approximation for wave slopes smaller than $ak<0.2$, even for Reynolds numbers of order 10^6. The deviation from the linear theory is also showed in Fig. 4 with the rate of stress $\tau_{TEO}/\tau_{DNS}$ as a function of $X\lambda$, where $\tau_{TEO}$ and $\tau_{DNS}$ are the integrated wall shear stresses obtained by theory [2] and DNS respectively (Eqns. 7 and 8), it can be seen that the aforementioned discrepancy is larger at smaller Reynolds numbers and bigger wave slopes.

\[
\tau_{TEO} = \int \tau_{wF}(x) dx / U_*,
\]

where $U_*$ is the average friction velocity given by $U_* = \sqrt{\tau_{av}/\rho}$. Here $\tau_{av}$ is the average shear stress along the wall. Nonlinear effects grow progressively as the wave slope $ak$ increases with a fixed Reynolds number, until a change in flow configuration is observed when the shear stress near the trough shifts sign as shown in the lower part of Fig. 2. In Fig. 2 and 3 the linear theory of Benjamin (equations 5.6 and 5.9 in his paper [2]) are also shown.

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\tau_{TEO} = \int \tau_{wF}(x) dx / U_* ,
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Figure 2: Wall dimensionless shearing stress using different wave slopes and Reynolds numbers. Solid line: linear theory; dotted line: DNS.

Figure 3: Wall normalized pressure using different wave slopes and Reynolds numbers. Solid line: linear theory; dotted line: DNS.

Figure 4: Ratio between theoretical and the simulated streamwise integrated stresses.

Figure 5: Vertical velocity profiles at the trough, for Re=10^4, for ak=0.35 (red), ak=0.2 (green), ak=0.1 (blue), ak=0.01 (magenta). Upper left: the circulation that occurs when ak=0.35. Lower right: recirculation when ak=0.2.

\[ \tau_{DNS} = \int \tau_{w0}(x) dx / u_* \quad , \]  

where \( \tau_{w0} \) and \( \tau_{w0} \) are the average shear stress along the wall (\( \tau_{w0} \)) obtained by the linear theory [2] and by the DNS respectively.
Finally, Table 1 records the appearance of the recirculation for different Reynolds numbers (columns) and wave slopes (rows). The symbol ‘x’ means there is recirculation, while the symbol ‘o’ means the contrary. The results of this table are represented also in Fig. 6 with the critical wave slope reported by Malevich [4], where the eddies appear.

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</table>

Table 1: Presence and absence of recirculation described by zeros and ones respectively at different numbers of reynolds and waveslopes.

![Figure 6: Presence of Eddies: ‘o’ no eddy; ‘x’ recirculation near trough. Continuous line: onset of recirculation from Malevich [4]; dotted line: critical convergence limit for [4].](image_url)

CONCLUSIONS

The validity range of linear theory for a Couette 2D flow over a wavy boundary was investigated using a numerical simulation code that solves the full 2D Navier-Stokes equations. Linear theory predictions were verified for the dual limit of large Reynolds numbers and small wave slope in the ranges 0≤ak≤0.3. The shear stress and pressure along the wavy wall were consistent with other results found in the literature. The linear approximation is accurate for wave slopes smaller than ak=0.2, with less than 5% of error in the shear stress. For Reynolds numbers larger than 3500 the range of validity of the approximation covers values of ak as large as 0.3, as long as the flow remains laminar. The onset of recirculation was also reported for different Reynolds numbers and wave slopes, which resulted systematically larger than the ones predicted by the theory of Malevich [4] for 2D channels, as shown in Fig. 6. The region in the ak-Re space covered by this study is partially outside the critical convergence area given by Malevich [4] as ak=Re^{1/2}.

PERSPECTIVES

Future work consists of simulating the 3D flow over a roughness change from wavy surfaces to flat terrain as occurs in coastal zones. This technique allows for the study of arbitrary geometries if different mappings are combined as is usually done with a Fourier series.

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NOMECLATURE

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REFERENCES


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