8 TURBULENT TRANSPORT

This section will provide an introduction to one of the most common approaches by which turbulence is treated and studied from a statistical viewpoint. This will serve as an introduction and background for some of the modeling approaches to be studied in the next section. We begin this discussion by applying the decomposition of the flow into its mean and fluctuating components as alluded to earlier. By applying this decomposition to the transport equations of momentum and kinetic energy we will be able to isolate some of the mechanisms by which the turbulence affects the mean flow. Mechanisms of energy exchange between the mean flow and the turbulence will be pointed out. Also, we can approximate the magnitude of terms appearing in these equations to determine which mechanisms are most important in certain flow regimes.

8.1 Reynolds Averaged Equations

To obtain an equation for the mean values of the dependent variables, we first decompose the variables into their mean and fluctuating components:

$$\begin{aligned} u_i &= \overline{u_i} + u_i' \\ p &= \bar{p} + p' \end{aligned} \tag{8.1}$$

Similarly, the stress tensor can be decomposed as

$$\sigma_{ij} = \overline{\sigma_{ij}} + \sigma_{ij}' \tag{8.2}$$

where

$$\overline{\sigma_{ij}} = \bar{p}\delta_{ij} + \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{8.3}$$

and

$$\sigma_{ij}' = p'\delta_{ij} + \mu\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \tag{8.4}$$

The mean and fluctuating rate of strain tensors are given by:

$$\frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) = \bar{S}_{ij} \tag{8.5}$$

and

$$\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) = s_{ij} \tag{8.6}$$

(For an incompressible fluid.) From the definition of our mean values we have

$$\overline{u_i'} = \overline{p'} = \overline{\sigma_{ij}'} = 0 \tag{8.7}$$

We then substitute Eq. 8.1 - 8.4 into the Navier-Stokes equations and average the entire equation. This gives

$$\rho\frac{\partial\overline{\bar{u}_i + u_i'}}{\partial t} + \rho\frac{\partial\overline{(\bar{u}_i + u_i')(\bar{u}_j + u_j')}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\overline{\sigma_{ij}} + \sigma_{ij}'}\right) \tag{8.8}$$

Applying Eq. 8.7 and noting that $\overline{\bar{u}_i} = \bar{u}_i$, yields the final equation for the average momentum transport:

$$\rho\frac{\partial\bar{u}_i}{\partial t} + \rho\frac{\partial(\bar{u}_i\bar{u}_j + \overline{u_i'u_j'})}{\partial x_j} = \frac{\partial}{\partial x_j}\overline{\sigma_{ij}} \tag{8.9}$$

Eq. 8.9 for the mean momentum resembles the full momentum equation except for the term $\partial/\partial x_j(\overline{u_i'u_j'})$. If u_i' and u_j' are uncorrelated, this term is zero. However, this is not generally the case. Physically, this correlation represents the mean value of the transport of fluctuating momentum by the fluctuating velocity field. From Eq. 8.9 it is clear that this transport of fluctuating momentum influences the transport of the mean momentum. This term can therefore be interpreted as a mechanism for momentum exchange between the mean flow and the turbulence. In other words, the velocity fluctuations produce an additional mean momentum flux that would not appear in laminar flow.

Momentum flux can be associated with a stress in the fluid. The turbulent momentum exchange mechanism appears in Eq. 8.9 is in the form of a divergence of a quantity we can thus associate with a stress. Turbulence can therefore be interpreted to produce additional stress in the fluid. As a result of this behavior, the correlation $\overline{u_i'u_j'}$ is called the *Reynolds stress tensor*, in honor of O. Reynolds who first applied this type of decomposition. With this interpretation for the velocity correlation, Eq. 8.9 can be rearranged and written as:

$$\rho\frac{\partial\bar{u}_i}{\partial t} + \rho\frac{\partial\bar{u}_i\bar{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\sigma_{ij}} - \overline{u_i'u_j'}\right) \tag{8.10}$$

Arranging the Reynolds stress tensor in the momentum equation in this manner also has bearing on how we will model the effects of turbulence a little later. The mean stress tensor in the turbulent flow can then be expressed as:

$$T_{ij} = \overline{\sigma_{ij}} - \overline{u_i'u_j'} = -\bar{p}\delta_{ij} + 2\nu S_{ij} - \overline{u_i'u_j'} \tag{8.11}$$

8.2 An Estimate For the Reynolds Stress

As you should now be becoming familiar with, dimensional analysis and scale analysis play an important role in the analysis of turbulent flows. In many cases, order of magnitude estimates are the only way of providing information about the turbulence terms, and play a major role in the formulation of turbulence models (to be discussed in soon). Let us assume the velocity fluctuations are characterized by $\sqrt{u'^2} \equiv u_t$. The integral scale of the flow (characterizing the large eddies) is given by L. With these scale relations, the mean stress tensor can be estimated as

$$\nu \bar{S}_{ij} \sim \nu \frac{u_t}{L} \tag{8.12}$$

and the Reynolds stress tensor estimated as u_t^2. The ratio of these two is

$$\frac{u_i' u_j'}{\nu \bar{S}_{ij}} \sim \frac{u_t^2}{\nu u_t / L} = \frac{u_t L}{\nu} = Re \tag{8.13}$$

The ratio of these two terms is proportional to the Reynolds number. Therefore, for high Reynolds number flow, we have $u_i' u_j' \gg \overline{\sigma_{ij}}$. This implies that viscous stresses have a lower order effect on the mean flow than the induced turbulent stresses. The viscosity thereby plays a small role in the transport of the mean momentum.

It is interesting to note that the Reynolds stress itself is usually a negative quantity if the mean velocity gradient, $\partial \bar{u}_1 / \partial x_2$, is greater than zero. In this case positive values of u_2' tend to convect fluid lower absolute values of u_1', resulting in a decrease in u_1', yielding a negative value for $\overline{u_1' u_2'}$.

In the introduction to this class we stated one of the characteristics of turbulent flow was the enhanced transport of mass momentum and energy. In the case of momentum transport we see this turbulent diffusivity results from the presence of the Reynolds stress tensor, which generally makes a much larger contribution to the overall momentum transport than the viscous stress tensor.

8.3 Turbulent Kinetic Energy

In § 8.1 the effects of turbulence on the mean momentum of the flow was illustrated. We would like to repeat that here for the kinetic energy. (We will soon see that the turbulent kinetic energy plays an important role in developing turbulence models. Also, it is useful to understand how the mean flow feeds kinetic energy to the turbulence.) For an incompressible fluid with constant transport coefficients, the transport equation for the kinetic energy is:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{u_i u_i}{2}\right) + \frac{\partial}{\partial x_j}\left(u_j \frac{u_i u_i}{2}\right) = & - \frac{1}{\rho}\frac{\partial u_j p}{\partial x_j} + \frac{\partial}{\partial x_j} u_i 2\nu S_{ij} \\ & - 2\nu S_{ij} \frac{\partial u_i}{\partial x_j} \end{aligned} \tag{8.14}$$

This can be derived directly from the Navier-Stokes equations with a little manipulation. The terms on the RHS of Eq. 8.14 represent the work done by pressure forces, the work done by viscous stresses, and the dissipation, respectively. This equation will sometimes be seen written as:

$$\frac{\partial}{\partial t}\left(\frac{u_i u_i}{2}\right) = - \frac{\partial}{\partial x_j}u_j\left(\frac{p}{\rho}+\frac{u_i u_i}{2}\right) + \frac{\partial}{\partial x_j}u_i 2\nu S_{ij} - 2\nu S_{ij}\frac{\partial u_i}{\partial x_j} \tag{8.15}$$

The first term on the RHS now represents the work done by the total dynamic pressure.

We next want to apply the Reynolds decomposition to Eq. 8.14. The decomposition of the kinetic energy gives:

$$u_i u_i = \bar{u}_i\bar{u}_i + 2\bar{u}_i u'_i + u'_i u'_i \tag{8.16}$$

Substituting Eq. 8.16 and Eqs. 8.1 - 8.7 in Eq. 8.14 and averaging the entire equation gives:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u'_i u'_i}\right) = &- \frac{\partial}{\partial x_i}\bar{u}_i\left(\frac{\bar{p}}{\rho}+\frac{1}{2}\bar{u}_j\bar{u}_j\right) + \nu\frac{\partial}{\partial x_i}\bar{u}_j 2\bar{S}_{ij} \\ &- \nu 2\bar{S}_{ij}\frac{\partial \bar{u}_j}{\partial x_i} - \frac{\partial}{\partial x_i}\overline{u'_i\left(\frac{p'}{\rho}+\frac{1}{2}u'_j u'_j\right)} \\ &- \frac{\partial}{\partial x_i}\bar{u}_j\overline{u'_i u'_j} - \frac{1}{2}\frac{\partial}{\partial x_i}\bar{u}_i\overline{u'_j u'_j} \\ &+ \nu\frac{\partial}{\partial x_i}\overline{u'_j 2s_{ij}} - \overline{\nu 2s_{ij}\frac{\partial u'_j}{\partial x_i}}\end{aligned} \tag{8.17}$$

This equation contains both the mean and turbulent kinetic energy. To obtain an equation for the mean turbulent kinetic energy equation alone we multiply Eq. 8.10 by $\bar{u}_i$. The resulting equation is:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right) + \frac{\partial}{\partial x_i}\bar{u}_i\left(\frac{\bar{p}}{\rho}+\frac{1}{2}\bar{u}_j\bar{u}_j\right) = &- \overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(-\overline{u'_i u'_j}\bar{u}_i) \\ &+ \nu\frac{\partial}{\partial x_i}\bar{u}_j 2\bar{S}_{ij} - \nu 2\bar{S}_{ij}\frac{\partial \bar{u}_i}{\partial x_j}\end{aligned} \tag{8.18}$$

The last term on the LHS of Eq. 8.18 and the last two terms on the RHS have the same physical interpretation as their counterparts in Eq. 8.14. As was the case in the equation for the mean momentum, there now appear additional terms due to the turbulence that effect the mean kinetic energy. The first term on the RHS of Eq. 8.18 represents the deformation work by the turbulent stresses. Because $\overline{u'_i u'_j}$ is usually negative, the action of this term tends to decrease, or take away kinetic energy from the mean motion. We will see shortly that this same term results in an increase in the turbulent kinetic energy. The second term on the RHS is the work by the turbulent stresses.

A similar order of magnitude scaling analysis as was done in **§8.1** for the Reynolds stresses shows that the viscous terms in Eq. 8.18 have a lower order effect than the other (turbulence) terms on the RHS of Eq. 8.18. Viscosity does not have a significant impact on the mean kinetic energy transport. This again shows that the large structures in turbulent flows are relatively independent of viscosity.

More information about the turbulence and its interaction with the mean flow can be obtained from the transport equation for the turbulent kinetic energy. This is now easily obtained by subtracting Eq. 8.18 from Eq. 8.17:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i'u_i'}\right)+\bar{u}_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'}\right) &= -\frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho}+\frac{1}{2}u_j'u_j'\right)}-\overline{u_i'u_j'}\frac{\partial\bar{u}_j}{\partial x_i}\\ &+\nu\frac{\partial}{\partial x_i}\overline{u_j'2s_{ij}}-\nu\overline{2s_{ij}\frac{\partial u_j'}{\partial x_i}} \qquad (8.19)\end{aligned}$$

The mechanisms that affect the turbulent kinetic energy on the RHS of Eq. 8.19 are the turbulent diffusion of mechanical energy, deformation work on the mean flow by the turbulent stresses, the viscous work by the turbulent shear stresses, and the viscous dissipation of the turbulent kinetic energy.

As opposed to the case for the mean flow kinetic energy equation, the viscous terms are important to the turbulent kinetic energy balance. The last term in Eq. 8.19 has a particular significance. It is always positive and therefore indicates a drain of energy. This is the viscous dissipation term.

For a statistically steady flow, the time derivatives in the equations for the mean and turbulent kinetic energy are zero. Also, $s_{ij}s_{ij} = s_{ij}\partial u_j'/\partial x_i$ (convince yourself this is so). Applying these two conditions, the equations for the mean and turbulent kinetic energy can be written as:

$$\begin{aligned}\bar{u}_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right) &= \frac{\partial}{\partial x_j}\left(-\frac{\bar{p}}{\rho}\bar{u}_j+2\nu\bar{u}_i\bar{S}_{ij}-\overline{u_i'u_j'}\bar{u}_i\right)\\ &+2\nu\bar{S}_{ij}\bar{S}_{ij}+\overline{u_j'u_i'}\bar{S}_{ij} \qquad (8.20)\end{aligned}$$

and

$$\begin{aligned}\bar{u}_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_j'}\right) &= -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j'p'}+\frac{1}{2}\overline{u_i'u_i'u_j'}-2\nu\overline{u_i's_{ij}}\right)\\ &-\overline{u_i'u_j'}\bar{S}_{ij}-2\nu\overline{s_{ij}s_{ij}} \qquad (8.21)\end{aligned}$$

These are the same equations as (3.1.11) and (3.2.1) given in Tennekes and Lumley[1]. Note that the deformation work term $-\overline{u_i'u_j'}\bar{S}_{ij}$ (or, equivalently $-\overline{u_i'u_j'}\frac{\partial\bar{u}_j}{\partial x_i}$) appears in both the equations for the mean and turbulent kinetic energy equation, but with an opposite sign. We also call this the production term as the effect of deformation work is to exchange energy between the mean flow and the turbulence. Usually, this term is negative (the Reynolds stress is usually negative) so that there is a net flow of kinetic energy from the mean flow to the turbulence. This is not, however strictly the case, as a *countergradient flux* of momentum and energy is observed in certain situations. In a purely laminar flow, viscous stresses dissipate the kinetic energy directly into heat. In turbulent flow, however, the deformation work caused by the turbulent stresses first converts mean flow kinetic energy into turbulent kinetic energy. It then cascades through the wavenumber spectrum (as explained in **§6**) before it is finally dissipated into heat.

The terms in parentheses on the RHS of Eq. 8.21 are in the form of a divergence of a vector quantity. These are *conservative* terms, and when integrated over an infinite flow field volume will vanish. These inertial terms do not create or destroy energy, they simply redistribute the turbulent kinetic energy from place to place in the flow. In wavenumber space (the Fourier transformed equations) the effect of theses nonlinear inertial terms is to redistribute the energy among the different frequencies. Without this, the energy in each mode would decay, independent of all other modes. Again, we see it is the nonlinearities that give turbulence its interesting characteristics.

In a statistically steady homogeneous turbulent flow a further simplification of Eq. 8.21 can be made. In this case, all spatial gradients of the turbulent transport quantities will be zero. Under this condition, the only remaining terms give:

$$-\overline{u_i'u_j'}\bar{S}_{ij} - 2\nu\overline{s_{ij}s_{ij}} = 0 \tag{8.22}$$

or

$$-\overline{u_i'u_j'}\bar{S}_{ij} = 2\nu\overline{s_{ij}s_{ij}} \tag{8.23}$$

In words, Eq. 8.23 is saying that the rate of production of the turbulent kinetic energy is balanced by the rate of viscous dissipation.

8.4 Turbulent Transport of a Passive Scalar

One of the main points of emphasis of this course is to study in detail the mixing of both passive and reacting scalars in turbulent flows. In particular we will be discussing the physical mechanisms acting in the overall mixing process, and discuss many of the methods used to model mixing. Subsequent notes will be devoted to this topic. For completeness of this section, however, a short discussion of the mean transport equation for a passive scalar in a turbulent flow is given here.

The "exact" equation for the transport of a scalar variable, ϕ is (assuming a constant diffusivity D):

$$\frac{\partial \phi}{\partial t} + \frac{\partial u_j \phi}{\partial x_j} = D\frac{\partial^2 \phi}{\partial x_j \partial x_j} \tag{8.24}$$

In Eq. 8.24 ϕ could correspond to any passive scalar such as temperature or a chemical species or contaminant in the flow with D as its appropriate diffusion coefficient. Decomposing the variables into their mean and fluctuating components and averaging the equation gives the transport for the mean value of ϕ.

$$\frac{\partial \bar{\phi}}{\partial t} + \frac{\partial \bar{u}_j \bar{\phi}}{\partial x_j} = -\frac{\partial \phi' u_j'}{\partial x_j} + D\frac{\partial^2 \phi}{\partial x_j \partial x_j} \tag{8.25}$$

or

$$\frac{\partial \bar{\phi}}{\partial t} + \frac{\partial \bar{u}_j \bar{\phi}}{\partial x_j} = \frac{\partial}{\partial x_j} \left(D \frac{\partial \bar{\phi}}{\partial x_j} - \overline{\phi' u'_j} \right) \tag{8.26}$$

Similar to the Reynolds stress term in Eq. 8.10, the correlation $\overline{\phi' u'_j}$ represents the turbulent transport of ϕ in the x_j direction. As written in Eq. 8.26 this can again be interpreted as an enhanced diffusivity due to the turbulence.

Note that in § 8.1 we did not derive an equation for the mean transport of fluctuating momentum as $\rho \overline{u'_i}$ is identically equal to zero. We can derive an equation for the *nonaveraged* fluctuating momentum equation by subtracting the averaged Navier-Stokes equation from the non-averaged equation. We will see that the resulting equation can be used to derive an equation for the dissipation rate (next section).