5 THE TRANSITION TO TURBULENCE

A study of turbulence is incomplete without some mention of fluid stability and the transition to turbulence. We have previously noted that transition occurs when the fluid equilibrium is upset. Although not a topic that will be explored in this class, here is a brief overview of some ideas and methods used in studying fluid stability.

The study of hydrodynamics stability is a complete area of study and specialization in itself. The main objectives of this branch of fluid mechanics are to understand the conditions under which a given laminar flow becomes unstable, and give some information about the subsequent development of the instability. The development of instabilities in a laminar flow is the first step towards the transition to turbulence. In most fluid mechanics applications the ability to control the transition would greatly increase engineering efficiency and performance. For example, to achieve the lowest drag around aerodynamic bodies such as aircraft or cars, it is most desirable to delay transition[1]Although in the case of airfoil design in extreme operating conditions “vortex generators” are sometimes used to promote a turbulent boundary layer on certain areas of the wing. This is because a turbulent boundary layer will not separate as easily as a laminar boundary layer in an adverse pressure gradient. There is a tradeoff here to obtain the best overall performance.. It is obviously important to know when and where transition occurs. In combustion devices, high turbulence levels promote the mixing of fuel and oxidizer. In supersonic combustion devices, there are difficulties encountered in obtaining well mixed reactants, as a result of the stability of these flows. A better understanding of the transition process is necessary to achieve better combustion performance.

Numerous excellent text books are completely devoted to hydrodynamic stability. See for example, Drazin and Reed[1], Chandrsekar[2], or Lin [3], to mention only a few. Here we will briefly touch upon some of the analytic methods, and the type of information that these analyses yield.

From a historical perspective, the most well known experiment on hydrodynamic stability was carried out by Osborne Reynolds in 1888. He performed a set of experiments in which he carefully studied the behavior of flow in a pipe by varying different flow conditions. Specifically, by varying the diameter of the pipe, the velocity of the fluid, and the viscosity of the fluid, Reynolds found that there was a relationship between these variables that indicated the transition from a smooth laminar flow, to a complex turbulent flow. Namely, when the value of VD/ν (which we now know as the Reynolds number) exceeded a particular value, the perturbations began to grow, and the instantaneous flow structure became very complex. The transition depended

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only on the value of the Reynolds number, and not on the values of the individual terms.

About the same time Reynolds was performing his careful experiments on the transition to turbulence in pipe flows, many other investigators were making progress in the study of hydrodynamic stability. Some of the important scientists studying this problem were Helmholtz, Rayleigh, Taylor, and Kelvin, among others. Many types of flow instabilities now carry their names.

Kelvin-Helmholtz Instability Kelvin-Helmholtz instability is the name given to instabilities that occur when two parallel fluid layers, each with a different velocity and density are in contact with each other. For certain values of the velocities and densities the interface between the two fluids will begin to oscillate, indicating the onset of instability. A special case of this is when the density of the two fluid layers is the same.

Taylor-Couette Taylor-Couette flow is the flow that occurs between two concentric cylinders when the outer cylinder is held fixed and the inner cylinder is rotating at some specified frequency. Many interesting phenomena can be observed in this type of flow. As the velocity of the inner cylinder increases, the flow becomes unstable and a new, qualitatively different steady flow arises. Toroidal vortices form down the length of the cylinder. As the velocity increases further, these toroidal vortices themselves become unstable. Eventually, a fully developed turbulent flow results. This type of fluid motion was first studied successfully by Taylor in 1923 assuming a cylinder of infinite length (thus neglecting end effects).

Rayleigh-Benard Convection Rayleigh convection refers to the fluid motion that develops when a stable fluid is heated from below. If we have a fluid of depth H with a higher bottom temperature than surface temperature, a lower density will develop at the bottom. If the temperature difference (density difference) becomes large enough an instability will develop. This particular configuration can also exhibit a sequence of transitions to other stable flow configurations before fully developed turbulence occurs. One of these quasi-steady states displays the formation of counter-rotating convection cells (Benard cells) throughout the fluid.

The Von-Karman Vortex Street Flow around a cylinder displays a very intricate vortex pattern as it becomes unstable in the form of a series of offset, counter-rotating vortex structures. This structure is called the Von Karman vortex street.

The above descriptions were given to provide a flavor of the various types of flow fields that can develop as a fluid undergoes transition to turbulence. A number of tools have been developed to study and predict the behavior of a fluid as it undergoes transition to turbulence. Three different analytic methods you should have some familiarity with are:

1. Linear (normal mode) analysis
2. Nonlinear Analysis
3. Dynamical Systems Approach

The normal mode approach is the oldest and most applied approach in the study of hydrodynamics stability. It has proven very useful in understanding the initial development of some rather complex flows. In the remainder of this section we will illustrate how this approach is used and what type of information it gives us.

5.1 Linear Analysis

The general idea behind linear analysis is to add a small perturbation to a given, steady flow field. Since the perturbations to the steady flow are assumed small, quadratic terms in the fluctuating variables are eliminated, resulting in a linear equation. The equations for these small perturbations are solved to see if they grow or decay with time. To illustrate this procedure, we first separate the dependent variables into their steady and fluctuating components:

$$u_i = u_{0i} + u_i' \tag{5.1}$$

$$p = p_o + p' \tag{5.2}$$

where u_{0i} is the solution to the steady flow, and u_i' are the perturbations. The expressions for the velocity and pressure are then inserted into the governing equations (mass and momentum). Neglecting the nonlinear terms, we are left with the following linear equations for the velocity perturbations,

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{5.3}$$

$$\frac{\partial u_i'}{\partial t} + u_{0j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial u_{0i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 u_i'}{\partial x_j \partial x_j} \tag{5.4}$$

The general approach to handling this equations is the method of "normal modes." Random disturbances to the flow field consist of a superposition of many modes. Because we have linearized the problem, each of these modes, if unstable, will grow at its own rate. The analysis then, consists of assuming a solution of the form $u_i = \sum a_i \exp(\omega_i t)$. This results in an eigenvalue problem for a_i, where ω_i are the eigenvalues. If the real part of ω_i is greater than zero, the perturbations will grow and the flow is considered unstable. The goal of this type of analysis is to determine the parameter range in which the flow becomes unstable, and to get some quantitative

Figure 5.1 Configuration for stability analysis of shear layer.

information about the instability. For example, the flow may be unstable to some perturbations, and stable for others. Also, the unstable modes will have various growth rates. This is the type of information we would like to obtain through our stability analysis.

As an example, let us consider the stability of the parallel shear layer we discussed above (Kelvin-Helmholtz instability). This flow is very unstable to any perturbations. Small disturbances can grow rapidly, leading to a complex flow structure. This particular flow has been studied extensively both in the laboratory, and numerically and analytically. It is a useful flow for analysis because such shear layers are approximated in many real flows, and under proper simplification, this is a flow that can be analyzed in detail. In Fig. 5.1 the configuration for this analysis is shown. The flow field consists of a mean velocity $-1/2\,U$ for $z > 0$, and $1/2\,U$ for $z < 0$. The position of the interface between the two surfaces is given by η. We will neglect viscosity and apply a small disturbance $\mathbf{u}$ to the basic flow. The perturbation velocities in the two streams will be identified by $\mathbf{u_1'}$ and $\mathbf{u_2'}$, where they each satisfy the incompressible Euler equations. The subscripts 1 and 2 refer to the upper and lower region of the domain shown in Fig. 5.1. In these two regions we can also assume the flow is irrotational. With this assumption the velocity can be expressed in terms of the gradient of a scalar function called the velocity potential:

$$\mathbf{u}_{1,2} = \nabla\Phi_{1,2} \tag{5.5}$$

Furthermore, the continuity equation for steady flow becomes:

$$\nabla^2\Phi_{1,2} = 0 \tag{5.6}$$

where Φ is the velocity potential.The boundary conditions for this problem are

$$\nabla\Phi_1 \to -\frac{1}{2}U \quad \text{as} \quad z \to \infty \tag{5.7}$$

$$\nabla\Phi_2 \to \frac{1}{2}U \quad \text{as} \quad z \to -\infty \tag{5.8}$$

$$\text{Dynamic B.C.} \quad p_1 = p_2 \quad \text{at} \quad z = \eta \tag{5.9}$$

$$\text{Dynamic B.C. :} \qquad \text{Verticalvelocitiesareequalacrosssurface} \tag{5.10}$$

The kinematic condition can be expressed as

$$w_1 = \frac{\partial\Phi_1}{\partial z} = \frac{\partial\eta}{\partial t} + \frac{\partial\Phi_1}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\Phi_1}{\partial y}\frac{\partial\eta}{\partial y}|_{z=\eta} \tag{5.11}$$

$$w_2 = \frac{\partial\Phi_2}{\partial z} = \frac{\partial\eta}{\partial t} + \frac{\partial\Phi_2}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\Phi_2}{\partial y}\frac{\partial\eta}{\partial y}|_{z=\eta} \tag{5.12}$$

By integrating Euler's equations for this flow we get

$$\frac{p_1}{\rho_1} + \frac{\partial\Phi_1}{\partial t} + \frac{1}{2}(\nabla\Phi_1)^2 + gz = C_1 \tag{5.13}$$

and

$$\frac{p_2}{\rho_2} + \frac{\partial\Phi_2}{\partial t} + \frac{1}{2}(\nabla\Phi_2)^2 + gz = C_2 \tag{5.14}$$

Applying the dynamic boundary condition at $z = \eta$ gives

$$\rho_1\left[C_1 - \frac{\partial\Phi_1}{\partial t} - \frac{1}{2}(\nabla\Phi_1)^2 - gz\right] = \rho_2\left[\frac{\partial\Phi_2}{\partial t} - \frac{1}{2}(\nabla\Phi_2)^2 - gz\right]\Big|_{z=\eta} \tag{5.15}$$

Evaluating the constants at steady state (this relationship must hold for the mean steady flow as well as the disturbed flow) gives:

$$C_1 = \frac{\rho_2}{\rho_1}\left(C_2 - \frac{1}{2}U_2^2\right) + \frac{1}{2}U_1^2 \tag{5.16}$$

Eqs. 5.5 - 5.15 describe the nonlinear stability of the interface. To look at the linear stability we consider a small perturbation to the mean state:

$$u_i = U + u_i' \tag{5.17}$$

or, in terms of the velocity potential,

$$\Phi_1 = U_1 x + \phi_1 \tag{5.18}$$

and

$$\Phi_2 = U_2 x + \phi_2 \tag{5.19}$$

Linearizing the kinematic boundary condition gives:

$$w_1\big|_\eta = \frac{\partial \Phi_1}{\partial z}\big|_\eta = \frac{\partial \eta}{\partial t} + \mathbf{u}_1 \cdot \nabla \eta = \frac{\partial \eta}{\partial t} \tag{5.20}$$

or

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \eta}{\partial t} = \frac{\partial \phi_2}{\partial z} \tag{5.21}$$

The linearized dynamic boundary condition is (using the given condition that $U_1 = -1/2U$ and $u_2 = 1/2U$):

$$\rho_1 \left(-\frac{1}{2} U \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial t} + g\eta \right) = \rho_2 \left(\frac{1}{2} U \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial t} + g\eta \right) \tag{5.22}$$

We also assume that the flow is periodic in space. The solution for the elevation and velocity potential can then be expressed as a superposition of normal modes.

$$\eta(x, y, t) = \sum_{l,m=0}^{\infty} A_{lm}(t) \exp[\sigma t + i(lx + my)] \tag{5.23}$$

$$\phi(x, y, t) = \sum_{l,m=0}^{\infty} B_{lm}(t) \exp(\mp kz) \exp[\sigma t + i(lx + my)] \tag{5.24}$$

These solutions are then substituted into the boundary conditions. The two kinematic boundary conditions give:

$$-kB_1 = \sigma A - \frac{1}{2} UiAl \tag{5.25}$$

and

$$-kB_2 = \sigma A + \frac{1}{2} UiAl \tag{5.26}$$

The dynamics boundary condition gives:

$$\rho_1 \left[-\sigma B_1 - gA + \frac{1}{2} UilB_1 \right] = \rho_2 \left[-\sigma B_2 - gA - \frac{1}{2} UilB_2 \right] \tag{5.27}$$

Eqs. 5.25 - 5.27 provide three equations for the three unknowns, B_1, B_2, and A. These equations can be written in matrix form as:

$$[M]\,[B] = 0 \tag{5.28}$$

where

$$[B] = \begin{bmatrix} A \\ B_1 \\ B_2 \end{bmatrix} \tag{5.29}$$

and

$$[M] = \begin{bmatrix} \sigma - \frac{1}{2}Uil & k & 0 \\ \sigma + \frac{1}{2}Uil & 0 & -k \\ g(\rho_2 - \rho_1) & -\rho_1(\sigma - \frac{1}{2}iUl) & \rho_2(\sigma + \frac{1}{2}iUl) \end{bmatrix} \tag{5.30}$$

For any solution to be possible we must have $\det M = 0$:

$$\det M = \rho_1 \left(\sigma - \frac{1}{2}Uil\right)^2 k + \rho_2 \left(\sigma + \frac{1}{2}Uil\right)^2 k + k^2 g(\rho_2 - \rho_1) = 0 \tag{5.31}$$

This equation gives a dispersion relation for σ:

$$\sigma = f(l, m, U, \rho_1, \rho_2) \tag{5.32}$$

In general, σ is a complex number, $\sigma = \sigma_r + i\sigma_I$, the real part giving the growth rate, and the imaginary part giving wave propagation information. If the real part is greater than zero we will get exponential growth and the flow is unstable. For the situation above, we get two modes:

$$\frac{\sigma}{kU} = -\frac{1}{2}i\frac{l}{k}\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \pm \left[\frac{l^2}{k^2}\frac{\rho_1\rho_2}{(\rho_1 + \rho_2)^2} - \frac{g(\rho_2 - \rho_1)}{kU^2(\rho_2 + \rho_1)}\right]^{\frac{1}{2}} \tag{5.33}$$

The flow will be neutrally stable ($\sigma_r = 0$) if the RHS of Eq. 5.33 is purely imaginary. This occurs for:

$$U^2 \leq \frac{(\rho_1 + \rho_2)}{\rho_1\rho_2}\frac{k}{l^2}g(\rho_2 - \rho_1) \tag{5.34}$$

The condition for instability is that the term in brackets in Eq. 5.33 is greater than zero ($\sigma_r > 0$):

$$U^2 > \frac{(\rho_1 + \rho_2)}{\rho_1\rho_2}\frac{k}{l^2}g(\rho_2 - \rho_1) \tag{5.35}$$

Note that σ_r can never be negative due to the square root in Eq.. 5.33. For a homogeneous fluid where $\rho_1 = \rho_2$ we have $\sigma = \pm\frac{1}{2}U$ so the flow is unstable to any disturbances. If $\rho_1 > \rho_2$ the flow is also unstable, whereas for $\rho_2 > \rho_1$, neutral stability is possible. However, for large enough l (short waves), the flow will always be unstable.

The example above illustrates the use of the normal mode approach for treating parallel flows. By assuming linear velocity profiles for the mean flow, more complex configurations can be studied by applying boundary conditions at each of the interface surfaces. For smoothly varying velocity profiles, the analysis of the differential equations is not so easy and the equations must be integrated numerically.

A more general approach for incompressible free shear flows is describe the velocity field by a stream function.

$$\frac{\partial \Psi}{\partial y} = u \tag{5.36}$$

$$\frac{\partial \Psi}{\partial x} = -v \tag{5.37}$$

where u and v are the two components of the velocity. It is next assumed that the stream function of the disturbance, ψ, can be given by:

$$\psi(x, y, t) = \phi(y) \exp[i(\alpha x - \beta t)] \tag{5.38}$$

where

$$\alpha = \frac{2\pi}{\lambda} \tag{5.39}$$

and

$$\beta = \beta_r + i\beta_I \tag{5.40}$$

Eq. 5.38 can also be written as:

$$\psi(x, y, t) = \phi(y) \exp[i\alpha(x - ct)] \tag{5.41}$$

In the above notation the sign of the imaginary part of c will then determine if the small perturbations will grow or decay with time. Inserting Eq. 5.40 for the stream function in the equation of motion (Eq. 5.4) yields the following equation for the amplitude of the stream function:

$$(U - c)(\phi'' - \alpha^2 \phi) - U'' \phi = -\frac{i}{\alpha \, Re}(\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi) \tag{5.42}$$

α and Re are assumed known. This is then an eigenvalue problem for the amplitude ϕ with the wave numbers c being the eigenvalues. This famous equation of boundary layer theory and hydrodynamic stability is called the *Orr-Sommerfield* equation and was first derived in 1907. It can generally be solved numerically to give information about when instability will set in. Fig. 5.2 shows a graphical representation of the solution to the above equation. Of particular interest is the region where c is greater than zero. In this region all perturbations will grow in time according to linear theory. Since all wave numbers are usually present in flows of practical interest, instability will generally set in when the critical Reynolds number is exceeded. The actual shape of Fig. 5.2 will of course be different for different flow configurations (flow over a flat plate, flow in a pipe, etc.)

The linear analysis discussed above is very useful in helping us identify flow regimes where instabilities may set in. Unfortunately, it does not tell the whole picture. Linear stability cannot answer questions regarding large amplitude disturbances, nor can it describe the finite amplitude, nonlinear flow that eventually occurs in the transition to turbulence. To address these issues, other approaches must be taken.

Figure 5.2 Stability region for boundary layer.

5.2 Non-Linear Analysis

As the linear instabilities develop, they will eventually undergo nonlinear interactions as the amplitudes of the unstable modes grow. Here we will very superficially mention some aspects of the nonlinear stability analysis. In the weakly nonlinear analysis we consider a stream function for the perturbation quantities of the form:

$$A(t)\psi(y)\exp[i\alpha(x-ct)] \tag{5.43}$$

The difference between this and the nonlinear case is the inclusion of a finite amplitude disturbance, $A(t)$, that takes into account the effects on stability that larger amplitude disturbances can have. When this expression is substituted into the equations of motion, the following equation for the amplitude is obtained:

$$\frac{d|A|^2}{dt} = \alpha c_i|A|^2 - a_1|A|^4 + \text{H.O.T}'\text{s} \tag{5.44}$$

Eq. 5.44 is known as the Landau equation and a_1 is the Landau constant. αc_i represents the amplifications factor of the linear analysis. Now depending on the value of a_1 the effect of a finite amplitude disturbance can either increase or decrease the growth rate of the linear modes. Eq. 5.44 can be converted to a linear equation for $|A|^{-2}$:

$$\frac{d|A|^{-2}}{dt} + \alpha c_i|A|^{-2} = a_1 \tag{5.45}$$

This equation has the solution:

$$|A|^{-2} = \frac{a_1}{2\alpha c_i} + \left(A_0^{-2} - \frac{a_1}{2\alpha c_i}\right)\exp(-2\alpha c_i t) \tag{5.46}$$

or

$$|A|^2 = \frac{A_0^2}{\frac{a_1}{2\alpha c_i}A_0^2 + \left(1 - \frac{a_1}{2\alpha c_i}A_0^2\right)\exp(-2\alpha c_i t)} \tag{5.47}$$

Consider first the case where αc_i is less than 0. In the linear case this would correspond to the case of exponential growth of the initial disturbance. However, in the weakly nonlinear analysis it can be shown that all perturbations will eventually arrive at some finite amplitude. If the initial perturbation amplitude is greater than this critical value it will decrease until this value is reached. This is the supercritical case. On the other hand, if the value of αc_i is positive, and if the perturbation amplitude is greater than a certain value, perturbations that would be expected to decay in the nonlinear analysis will grow if a_1 is less than zero. The amplitude of the initial disturbances can clearly effect the development of the flow.

5.3 Dynamical Systems

Another approach to studying hydrodynamic stability that has attracted recent attention is the application of the theory of dynamical systems. To introduce this idea it is informative to consider two different ideas concerning the transition to turbulence in hydrodynamic systems. These are the Landau-Hopf and Ruelle-Takens theories on transition. In the Landau-Hopf theory, the transition is seen as a series of bifurcations that lead to an increasingly more complex flow. As the Reynolds number approaches a critical value for the flow under consideration, a periodic flow develops. As the Reynolds number is increased further, this periodic flow itself becomes unstable, giving rise to additional periodic components of the flow. This process continues until a very complex, quasi-periodic flow develops.

In the Ruelle-Takens theory, totally chaotic motions are assumed to arise after only a few bifurcations. Beyond this point the dynamics of the flow are considered inherently chaotic. An important distinction between the quasi-periodic and chaotic flows has to do with their dependence on initial conditions. Chaotic (although deterministic) motion is very sensitive on the initial conditions. Two identical flow fields with only infinitesimal differences in their initial states can exhibit solutions that rapidly diverge from one another. In quasi-periodic flow, however, a slight change in the initial conditions would not be expected to have a great effect on the subsequent fluid motions. Spectral analysis of turbulent flows appears to support the ideas of Ruelle and Takens.

The dynamical systems approach to studying fluid stability began with the discovery that very simple, low order nonlinear equations can show remarkable properties of chaotic motions that bear a resemblance to fluid turbulence. In 1963, Lorentz[4] obtained a set of ordinary differential equations by severely truncating a set of equations that describe thermal convection. Although having much fewer degrees of freedom than real turbulence, these equations were parameterized by the important nondimensional numbers that characterize Rayleigh convection. By varying these parameters in the truncated system of equations, the bifurcation of the solutions to other periodic solutions, and then totally chaotic motion could be studied.

Although the equations are deterministic, the development of a physical system governed by them is essentially unpredictable. This is because of their extreme sensitivity to initial conditions. Real phenomena, like the weather, also appear to have this character. This is why it is essentially impossible to predict the weather more than a few days in advance. Just pay attention to the weather reports for a while before you debate this.

References

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