4 STATISTICS OF THE FLOW

Owing to the complexity of turbulent flows, they are generally studied using tools of statistical analysis. For engineering design purposes there is no conceptual problem with this, as it is seldom necessary to know all the details of the turbulent velocity field. The efficiency of a device such as a heat exchanger, mixing vessel, or combustor will be measured by its average performance over time. For most engineering purposes we therefore generally neglect the details of the turbulence and work only with averages.

From the point of view of gaining physical understanding of turbulent flows, it is also often convenient to describe the flow in terms of it's statistical properties. A description of the flow in wavenumber space is also often more informative than in terms of raw data. In the following, we will define and interpret some of the more commonly used single and multi-point statistics and use these statistics to describe different aspects of the flow.

4.1 Mean Values

The simplest statistical property is the mean, or first moment. In section **??** we defined the expectation of a random variable and the higher-order single point moments by using the probability density function. In laboratory experiments or in the use of numerically obtained data, the mean value of a random variable at a particular spatial location can be obtained by averaging the long time measurement of that variable:

$$\overline{\phi} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \phi(t)dt \tag{4.1}$$

where $\overline{\phi}$ indicates the mean value of the random variable ϕ. This time average only makes sense if Eq. 4.1 is independent of t_0 and independent of T for large T (i.e., the integral converges). In such a case we would call the flow a "statistically steady," or a "stationary" process.

In a flow configuration where Eq 4.1 does not converge, either an "ensemble" or "volume" average must be used to describe the mean flow behavior. The volume average is defined as:

$$\overline{\phi} = \lim_{vol\to\infty} \frac{1}{vol} \int_{vol} \phi(\mathbf{x})dxdydz \tag{4.2}$$

where the integration is now performed over a volume, vol, at one instant in time. Equation 4.2 only makes sense when the statistical properties do not depend on spatial position. The ensemble average is defined as:

$$\overline{\phi} = \lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \phi(x) \tag{4.3}$$

In Eq. 4.3 the summation is over a number of samples, N, taken at the same location x, at the same time, t, for N different realizations, or experiments. For a stationary process, the averaging defined by Eq. 4.1 and 4.3 are the same. This is called the "ergodic hypothesis."

4.2 Higher Order Moments

The n^{th} order central moment, defined earlier using the pdf, can also be computed from

$$\overline{(\phi - \overline{\phi})^n} \tag{4.4}$$

Assuming, of course, that the averaging process indicated by the overline is properly defined and converges.

A hierarchy of moments can be defined which describe the statistical state of a random variable (the velocity field being one of the random variables in a turbulent flow). Mathematically, the turbulent velocity field is generally treated as a random variable with a mean and fluctuating component. Letting ϕ' represent the fluctuating component of the random variable, the random variable ϕ can be expressed as

$$\phi = \overline{\phi} + \phi' \tag{4.5}$$

or, for the velocity field

$$u_i = \overline{u_i} + u_i' \tag{4.6}$$

After the mean value, the next most important statistical property of turbulence is its second central moment, or variance.

$$\text{var}(\phi) = \overline{(\phi - \overline{\phi})^2} \tag{4.7}$$

This measure of a random variable gives a quantitative measure of how large the variations from the mean value can be expected to be. In the terminology of turbulence, we often speak of the *turbulence intensity*, which is just the square root of the second central moment, or the root mean square of the velocity fluctuation, $\sqrt{\overline{u'^2}}$. As it's name implies, the value of this term gives a measure of the intensity of the turbulence.

The third central moment is called the *skewness*. This gives a measure of the symmetry of the probability distribution (see below) of the random variable. For a perfectly symmetric distribution, the skewness is zero. Continuing this hierarchy, the fourth central moment, or kurtosis, gives a measure of how fast the probability distribution goes to zero. A large kurtosis would indicate that values of the random variable, far from the mean value exist with higher probability than lower kurtosis functions.

4.3 Two-Point, Time Statistics

The moments defined above are single point moments. That is, they contain only information about a random variable at a point. In a turbulent flow, it is important to have some statistical measure of spatial information about the flow. For example, to draw conclusions about length scale information, two point statistics are needed. The autocorrelation function is the correlation between velocity components at two different times, defined by $\overline{u(t)u(t+\tau)}$. The normalized correlation function, the *autocorrelation coefficient*,$\rho(\tau)$ is defined as

$$\rho(\tau) = \frac{\overline{u(t)u(t+\tau)}}{\overline{u^2(t)}} \tag{4.8}$$

Note that $\rho(0) = 1$. Also, from Schwartz's inequality, $\rho(\tau) < 1$ for all $\tau \neq 0$.

The correlation tensor is often used to define an *integral scale* of turbulence:

$$L_v = \int_0^\infty \rho(\tau)\, d\tau \tag{4.9}$$

L_v gives an estimate of the time interval over which the velocity component u is correlated.

The spatial correlation tensor, R_{ij} gives the correlation between velocity components at two different spatial locations and has an important interpretation in turbulent flows. It is defined by

$$R_{ij}(\mathbf{r}) = \overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})} \tag{4.10}$$

To describe the various scales of spatial motion in a turbulent flow it is more instructive to work with the Fourier transform of the correlation tensor rather than the correlation tensor itself.

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^2} \iiint R_{ij} \exp(-i\mathbf{k}\cdot\mathbf{r})d\mathbf{r} \tag{4.11}$$

$\Phi_{ij}(\mathbf{k})$ is appropriately called the *spectrum tensor* or *spectral density* as it represents the contribution of a wavenumber, k, to the value of R_{ij}. In other words, $\Phi_{ij}(\mathbf{k})$ gives wavenumber distribution of the correlation tensor. Each wavenumber k corresponds to a physical space structure with a wavelength of $2\pi/k$.By use of the inverse transform we have:

$$R_{ij}(\mathbf{r}) = \iiint \Phi_{ij} \exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k} \tag{4.12}$$

Of particular significance is the sum of the diagonal components, $R_{ii}(\mathbf{r})$, for $\mathbf{r} = 0$. For this case we have

$$R_{ii}(0) = u_1^2 + u_2^2 + u_3^2 \tag{4.13}$$

which is twice the kinetic energy. Setting $\mathbf{r} = 0$, we can write Eq. 4.12 as

$$\begin{aligned} \frac{1}{2}R_{ii}(0) &= \frac{1}{2}\iiint \Phi(\mathbf{k})d\mathbf{k} \\ &= \int_0^\infty \left[\frac{1}{2}\iint \Phi_{ii}(\mathbf{k})d\sigma\right] dk \\ &= \int_0^\infty E(k)dk \end{aligned} \tag{4.14}$$

where

$$E(k) = \frac{1}{2}\iint \Phi_{ii}(\mathbf{k})d\sigma \tag{4.15}$$

$E(k)dk$ represents the contribution to the kinetic energy at a wavenumber of k in a spherical shell, σ, of thickness dk and is called the three dimensional energy spectrum. Integration of $E(k)$ over all k gives the total kinetic energy. With an eddy of a particular size, l, associated with a wavenumber of certain magnitude, k, the energy spectrum, $E(k)$, can be interpreted to give the distribution of energy among the different eddy sizes. As discussed above, the contribution of a wavenumber k corresponds to a structure with a wavelength of $2\pi/k$. The energy of and eddy of size $2\pi/k$ is therefore proportional to $kE(k)$. A large portion of the theoretical work on turbulent flows (including modeling) is concerned with the description of energy in the wavenumber spectrum and the transfer of energy among the different wavenumbers and frequencies.

4.4 Homogeneous and Isotropic Turbulence

A simplification in the mathematical treatment of turbulence comes about if we consider a flow in which the statistical quantities are independent of space. A turbulent flow with this property is call "homogeneous". In a homogeneous turbulent flow, the correlation tensor given by Eq. 4.10 is independent of position, $\mathbf{x}$, but still depends on the vector $\mathbf{r}$. If, in addition, the statistics are independent of orientation, the flow is considered "isotropic". In an isotropic flow, the correlation tensor depends only on the magnitude of r, and is independent of direction.

Most of the turbulent flows we are interested in are neither completely isotropic or homogeneous. However, much of the analysis and theory of turbulence has been formulated for isotropic turbulence. This results primarily from the complexity of the nonlinear governing equations which prevents detailed analysis to be performed. In the case of homogeneous isotropic turbulence, the equations simplify significantly. As a result, these simplified flows have been studied in some detail. Fortunately, though, there are regimes within the turbulence spectrum where it becomes increasingly likely that the flow can be treated as homogeneous and isotropic. In the next section (5) we will discuss how the turbulence structure is affected by vortex stretching. As the vortex structures (eddies) are broken down to smaller scales, and as the time scales associated with these structures decrease, a loss of any preferred orientations to mean shear is expected. At small enough length scales, it is then often assumed that the turbulence is isotropic. This is very important for many of the turbulence closures which are presently being studied. Some of the implications of this will be pointed out later.

The terminology *fully developed* turbulence is sometimes used to describe the state of the flow at length scales small enough that the information about the large scale motions or energy input mechanisms have been lost.

Turbulent flows are always unsteady and the flow at a particular point fluctuates intensely. However, the statistical behavior of a turbulent flow often does not change appreciably with time. A *statistically steady* or *stationary* turbulent flow is one who's statistics are approximately constant in time.