1   INTRODUCTION

It is reasonable to describe turbulence in fluids as "random" or "chaotic" behavior (motion) of the fluid that cannot be described exactly. The majority of flows found in nature and in engineering applications are turbulent. Turbulence has a large influence on the transport properties of the flow and other engineering applications, and is thus a subject that has drawn a significant amount of attention. It may therefore seem surprising, that although most flows of practical engineering importance are turbulent, a rigorous definition of turbulence is difficult to formalize. It is sometimes even difficult to agree, particularly when discussing two-dimensional or low-Reynolds number flows, as to whether a particular flow is turbulent or not. Fluid turbulence can be an intimidating subject to get involved with. Discussion range from the very mathematical, to the very qualitative and conceptual - with plenty of disagreement among the "experts".

Despite this, several definitions of turbulence have been formulated. Some are:

"Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquid, when they flow past solid surfaces or even when neighboring streams of the same fluid flow past or above one another." (G.I. Taylor quoted from von Karman.[1])

"Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned. (Hinze, [2].)

Turbulence is a three-dimensional time-dependent motion in which vortex stretching causes velocity fluctuations to spread to all wavelengths between a minimum determined by the viscous forces and a maximum determined by the boundary conditions of the flow. It is the usual state of fluid motion except at small Reynolds numbers." (Bradshaw, *An Introduction to Turbulence and its Measurement.")*

The study of turbulence can lead to emotional and tense debates. These result from different schools of thought and different emphasis on what is important. Although many of the ideas which lay the foundation for much theoretical and modeling work are not new, advances in modeling and understanding are continually being made, helping to keep debate alive.

In this course we will begin by discussing some general features concerning the nature and physics of turbulent flow. We will first approach this on a very qualitative level where some of the important physical and statistical properties of turbulence will be introduced. A large part of the class will be devoted to studying and discussing models of turbulent *mixing* processes. It will be important to understand exactly what we mean by *mixing* and the distinction we make between the effects of turbulence on *momentum* transport and on *scalar mixing*. These ideas will be emphasized

throughout the class.

We will treat turbulence as a continuum phenomenon - meaning that the length and time scales of the flow field are significantly larger than the length and time scales that describe the microscopic molecular processes. As such, it is widely agreed that the Navier-Stokes equations, along with appropriate equations expressing conservation of mass and energy provide an exact description of the turbulent flow field. When boundary and initial conditions are properly specified, these equations provide an exact description of the flow. In practice, it is not possible to generate solutions to these equations as they are a set of simultaneous, nonlinear partial differential equations to which no solution has been found for the general case. These equations are, however, the starting point for most analytical and numerical approaches to treating turbulence. It is assumed for this course that you are familiar with these equations. Although exact solutions are rare, you should feel comfortable with the physical interpretation of the terms in these equations. After a review of the mathematical notation we will be using, a brief derivation and review of the equations will be provided.

These notes are a supplement to material presented in class and reading out of the assigned text. No single text book can cover the range of current research topics and applications in turbulent flows. The assigned text for this class is "Turbulent Flows," by S. B. Pope.[3]. Several texts, collections, and individual papers must be consulted to obtain an in-depth view of any particular area. Throughout the course a list of such references will be provided. These include some classical texts such as Tennekes and Lumley,[4] and Hinze.[2] Some texts focusing on modeling for applications in CFD include Wilcox[5] and Rodi[6].

1.1 Tensor Notation

To deal with the mathematics associated with turbulent flow, a reasonable working knowledge of Cartesian tensors is necessary. Although this can get to be a pretty messy topic, what you need to know in order to work the equations and relationships encountered in fluid mechanics is pretty easy. Some of the notation is summarized below.

A vector is a quantity having both magnitude and direction. One way of representing a vector or a vector operation is by the use of "Einstein" index notation. For example, the velocity vector, $\mathbf{V} = \vec{v} = (u, v, w)$ can be written as $(u_1, u_2, u_3) = u_i$. Similarly, the coordinate vector $\mathbf{x}$ or $\vec{x}$ can be written as $x_i = (x_1, x_2, x_3)$. This illustrates the first property of index notation: *if an index (say, i) appears only once in a term, it is a "free" index and can represent any allowed value for that index.*

A commonly encountered vector quantity is the gradient of a scalar. This is written as

$$\nabla \phi = (\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}) \qquad (1.1)$$

In tensor notation this is

$$\nabla \phi = \frac{\partial \phi}{\partial x_i} \tag{1.2}$$

It is important to keep in mind the physical interpretation of these symbols. The gradient simply represents the rate of change of some scalar property (like temperature, pressure, chemical species concentration, etc.) in space. Since it is a vector, the gradient has, of course, both magnitude and direction. Large values represent rapid changes, while small values indicate fairly uniform conditions.

Another property of tensor notation is that repeated indices represent summation over that index. For example the vector dot (or scalar) product is defined as

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i \tag{1.3}$$

Similarly the divergence of a vector is defined as

$$\nabla \cdot \mathbf{A} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \frac{\partial a_i}{\partial x_i} \tag{1.4}$$

The divergence of a vector has a particularly significant importance in fluid dynamics. The divergence of the velocity field, $\frac{\partial u_i}{\partial x_i}$, represents the volumetric rate of change of a fixed mass of fluid. For an incompressible fluid, $\nabla \cdot \mathbf{V} = 0$.

A tensor quantity, like the strain rate would be represented by a double index:

$$S_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{1.5}$$

Two commonly used tensors with special significance are the Kronecker delta tensor, δ_{ij}, and the permutation tensor, ϵ_{ijk} These tensor quantities are defined by:

$$\begin{aligned} \delta_{ij} &= 1, & \text{if } i = j, \\ &= 0, & \text{if } i \neq j. \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} \epsilon_{ijk} &= 0, & \text{if any } i, j, k \text{ are equal} \\ &= 1, & \text{if } \quad i, j, k \text{ are cyclic clockwise} \\ &= -1, & \text{if } \quad i, j, k \text{ are cyclic counterclockwise} \end{aligned} \tag{1.7}$$

As an example, $\epsilon_{11k} = 0$ for any k, $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$. The delta tensor is also referred to as the "substitution" tensor as a result of the following property: $a_i \delta_{ij} = a_j$. In general, in an expression operated on by δ_{ij}, the effect is to replace any occurance of i by j (or vice versa). This also is useful in manipulating formulae in tensor notation. Another useful property is the following relationship between the delta tensors and the permutation tensor:

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \tag{1.8}$$

This is also very useful in manipulating formula written in tensor notation.

The use of the permutation tensor makes for a compact notation for expressing the vector cross product:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_i & A_j & A_k \\ B_i & B_j & B_k \end{vmatrix} = \mathbf{i}(A_2 B_3 - A_3 B_2) + \dots \tag{1.9}$$

Using tensor notation and remembering repeated indices indicate summation over that index, Eq. 1.9 can be written as

$$C_i = \epsilon_{ijk} A_j B_k \tag{1.10}$$

The vorticity vector, ω_i, is defined as the curl of the velocity field. Using vector symbols and tensor notation it can be written in either of the two following forms:

$$\omega = \nabla \times \mathbf{V} \tag{1.11}$$

or

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \tag{1.12}$$

Convince yourself that this is so.

This is basically all you will need to know. The notation throughout the class will be somewhat inconsistent in that we will switch back and forth between index notation and the use of symbolic operators (∇, $\nabla\cdot$, $\nabla\times$, etc.). This is just because we will use whatever comes easiest for any particular case. Make sure and familiarize yourself with the equivalent notations given in Eqs. (1.9, 1.10 1.11 and 1.12).

1.1.1 Symmetric and Antisymmetric Tensors

Recall that any second order tensor can be decomposed into a symmetric and antisymmetric component. For example, take the velocity gradient tensor, $\frac{\partial u_i}{\partial x_j}$. It's symmetric S_{ij} and antisymmetric R_{ij} components are:

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{1.13}$$

$$R_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \tag{1.14}$$

S_{ij} is the *rate of strain* tensor and R_{ij} is the rotation tensor. Both will play roles in subsequent developments.

By their nature, the tensor product of a symmetric and antisymmetric tensor is zero:

$$S_{ij} R_{ij} = 0 \tag{1.15}$$

By this definition note that

$$\frac{\partial u_i}{\partial x_j} S_{ij} = S_{ij} S_{ij} \tag{1.16}$$

(Show this)

1.2   Scale Analysis

Since exact solutions to the governing equations of fluid motion exist only in the most simplified cases, various types of approximate analysis is used to study the behavior of turbulent flow. In particular, many relationships can be derived based on order of magnitude estimates. These types of relationships will prove extremely useful in both the interpretation of turbulence phenomena and in the development of models to describe the effects of turbulence. The analysis used to generate the approximate relationships is termed scale analysis. Below we first illustrate its use by an example, then set down a few rules for its application.

A typical example of the use of scale analysis is in determining the time for a point source of a particular chemical constituent to diffuse across a certain distance. (This example and the following discussion is based on a presentation of this by Bejan.[7].

Consider a room with a linear dimension L. At one end of the room, a chemical with molecular diffusivity D is released. We wish to use scale analysis to determine the time for the chemical to diffuse across the room. The equation governing this process is the diffusion equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x_i \partial x_i} \tag{1.17}$$

The first step in the scale analysis process is to estimate the order of magnitude of the terms in Eq. 1.17. First we write

$$\frac{\partial C}{\partial t} \sim \frac{\Delta C}{t_o} \tag{1.18}$$

This expression can be interpreted as "the scale over which the concentration changes in a time t is ΔC. The notation $\sim$ can be read as "scales as" or "scales like," or "is of the same order of magnitude." Similarly, the rhs of Eq. 1.17 can be expressed as

$$D\frac{\partial^2 C}{\partial x_i \partial x_i} \sim D\frac{\Delta C}{L^2} \tag{1.19}$$

If Eq. 1.17 holds, the lhs is equal to the rhs, therefore we have:

$$\frac{\Delta C}{t_o} \sim D\frac{\Delta C}{L^2} \tag{1.20}$$

or

$$t_o \sim \frac{L^2}{D} \tag{1.21}$$

This result will compare well with exact solutions. Although we are not able to get quantitative results with this type of approximate analysis, the trends are correct - if the analysis is applied correctly. In applying this analysis, the following must be kept in mind:

1. Carefully define your spatial domain over which your analysis is to be performed. If you have spatial derivatives in an equation to which you wish to apply scale analysis, clearly state the domain over which that change occurs. On the other hand, if a time scale is given, make sure to carefully specify that. Each application will be different, so carefully specifying these parameters is important. You will see many different examples of this in these notes.

2. Consider an equation that consists of several terms:

$$A + B = C + D \tag{1.22}$$

   In this case we obviously have lhs $\sim$ rhs. Now if $o(A) > o(B)$ and $o(C) > o(D)$ then $A \sim C$. In other words, the order of magnitude of a sum is the order of magnitude of the dominant term. The notation $o(A)$ reads "the order of magnitude of A" and $o(A) > o(B)$ reads "the order of magnitude of A is greater than the order of magnitude of B."

3. For a product, $A = BC$, the order of magnitude is equal to the product of the order of magnitude of the individual terms:

$$o(A) \sim o(B)o(C). \tag{1.23}$$

   Similarly, for a quotient, we have $o(A) \sim o(B/C) \sim o(B)/o(C)$.

1.3 The Probability Density Function

Since turbulence is often characterized as "random" fluid motions, it is often most useful to characterize a turbulent flow by its statistics, rather than by it's detailed instantaneous structure. In a short while, we will discuss some of the many different statistical properties of turbulent flow. The purpose of this section is simply to introduce the idea of the probability density function.

When treating any function as a random variable, its value can only be specified with a certain probability. The complete statistical description of a random variable is given by its *probability distribution* at n points of space-time. The one point *probability density function*, or pdf, of a random variable ϕ provides the complete statistics of a random variable at an individual point in space. It is defined as

$$p_\phi(x)dx = \text{probability that } \phi \text{ has a value between } x \text{ and } x + dx \tag{1.24}$$

An immediate result of this definition is

$$\int_{-\infty}^{\infty} p(x)\, dx = 1 \tag{1.25}$$

since the probability of ϕ taking on some value between $-\infty$ and ∞ is 1. Since this is a one point pdf, multi-point correlations of information regarding length-scale information cannot be obtained at this level of description.

Figure 1.1 a. A hypothetical probability density distribution of the random variable ϕ. b. The cumulative distribution of ϕ.

The mean or expectation of the random variable is easily expressed in terms of the pdf:

$$\overline{\phi(x)} = \int_{-\infty}^{\infty} x p(x) dx \tag{1.26}$$

while the second central moment is

$$\text{var}(\phi) = \int_{-\infty}^{\infty} (x - \overline{\phi})^2 p(x) dx \tag{1.27}$$

A hierarchy of higher order moments and second moments can be similarly defined.
The cumulative distribution is defined as

$$P(x) = \int_{-\infty}^{x} p(x)\, dx \tag{1.28}$$

(Note that we are using the lowercase letters to denote the pdf, and upper case letters to denote the cumulative distribution.) $P(x)$ is a monotonically increasing function of x. From the definition of $p(x)$, we have $P(-\infty) = 0$ and $P(\infty) = 1$. In Fig. 1.1 the pdf and cumulative are illustrated schematically.

The cumulative distribution will prove to be useful later in the course when we wish to generate a series of random numbers that satisfy a particular pdf. Briefly, this is done as follows: Given a pdf, compute the cdf as described above. If the cdf can be expressed analytically, express the random variable ϕ as a function of the cdf. Then by choosing a random number uniformly distributed between $P(-\infty)$

REFERENCES   8

and $P(\infty)$, and using it in the resulting expression for $\phi = f(P(x))$, a sequence of random numbers can be generated to give a random variable ϕ that satisfies $p_\phi(x)$. If an analytic expression for $P(x)$ cannot be obtained, the procedure to generate the random variable can be accomplished numerically.

For a more thorough discussion, see the text (Pope, pp. 37–53).

References

[1] T. von Karman. Turbulence. *Journal of the Aeronautical Sciences*, 41:1109, 1937.

[2] J. O. Hinze. *Turbulence.* McGraw-Hill, New York, 1959.

[3] S.B. Pope. *Turbulent Flows.* Cambridge University Press, Cambridge, 2000.

[4] H. Tennekes and J. L. Lumley. *A First Course in Turbulence.* Massachusetts Institute of Technology Press, Cambridge, 1972.

[5] D.C. Wilcox. *Turbulence Modeling for CFD.* DWC INdustries, Inc., La Canada, CA, 1998.

[6] W. Rodi. *Turbulence models and their application in hydraulics.* 1979.

[7] Bejan. *Convection.* Wiley.