# Unsteady diffusion

We'll discretize the unsteady diffusion equation

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$

over a 1D control volume (see Fig. 1)

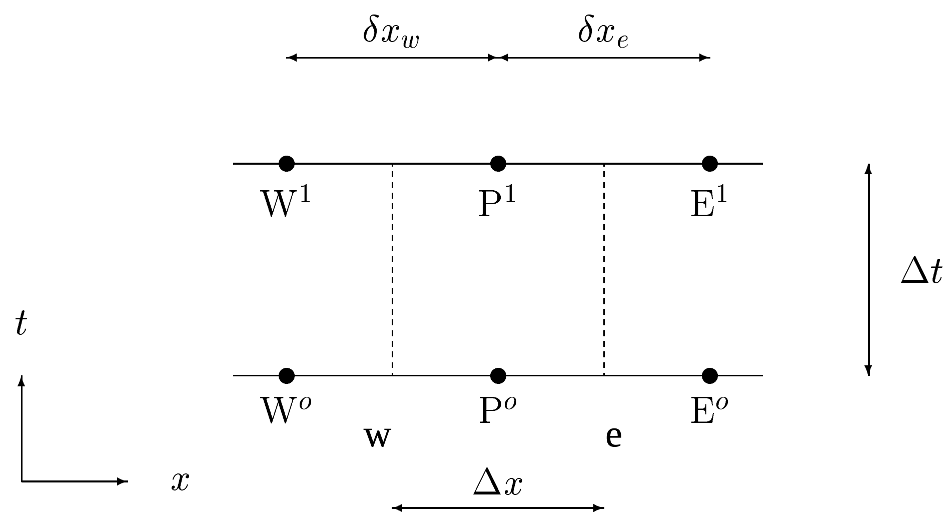


Figure 1. 1D control volume

We integrate in space and time

$$\int_t^{t+\Delta t}\int_w^e \rho c_p \frac{\partial T}{\partial t} dx dt = \int_t^{t+\Delta t}\int_w^e \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) dx dt$$

<u>LHS:</u>

$$\int_w^e \rho c_p \left[\underbrace{T^1}_{t+\Delta t} - \underbrace{T^o}_{t}\right] dx = \rho c_p (T_P^1 - T_P^o)\Delta x$$

RHS:

$$\int_t^{t+\Delta t}\left[\left(k\frac{\partial T}{\partial x}\right)_e-\left(k\frac{\partial T}{\partial x}\right)_w\right]dt =$$
$$\int_t^{t+\Delta t}\left[k_e\frac{T_E-T_P}{\delta x_e}-k_w\frac{T_P-T_W}{\delta x_w}\right]dt$$

At what time should $T_W$, $T_P$ and $T_E$ be taken?

1. Fully implicit: take them at the new time step $t+\Delta t$, i.e. $T_W^1$, $T_P^1$ and $T_E^1$ (first-order accurate).
2. Fully explicit: take them at the old time step $t$, i.e. $T_W^o$, $T_P^o$ and $T_E^o$ (first-order accurate).
3. Use central differencing in time (Crank-Nicolson). Second-order accurate. Note that this is what we did in space when integrating the LHS.

- Fully explicit (cf. Eq. 8.12 in M & V)

$$a_P T_P = \underbrace{a_E T_E^o + a_W T_W^o + \left[a_P^o - (a_W + a_E)\right] T_P^o}_{S_U}$$
$$a_E^o = \frac{k_e}{\delta x_e},\; a_W^o = \frac{k_w}{\delta x_w},\; a_P^o = \frac{\rho c_p \Delta x}{\Delta t}$$
$$a_P = a_P^o$$

No superscript means new time level, i.e. $T_P = T_P^1$ etc. The explicit scheme is stable only for time steps for which $\Delta t < \rho c_p(\Delta x)^2/2k$ is satisfied. This is a severe restriction which means that only very small time steps are allowed. The advantage is that each time step is very cheap. As seen from the above equation, we don't need to iterate since the right-hand side is known from the old time step: hence, the name ”explicit”. Note that the scheme is only first-order accurate in time (cf. with upwind scheme in space).

- Fully implicit (cf. Eq. 8.16 in M & V)

$$a_P T_P = a_E T_E + a_W T_W + \underbrace{a_P^o T_P^o}_{S_U}$$

$$a_E = \frac{k_e}{\delta x_e},\ a_W = \frac{k_w}{\delta x_w},\ a_P^o = \frac{\rho c_p \Delta x}{\Delta t}$$

$$a_P = a_E + a_W + a_P^o$$

For this scheme we must iterate since the right-hand side is unknown (the equation is *implicit*). The advantage is that the scheme is unconditionally stable, which means that very large time steps can be taken. However, as in space, large (time) cells (i.e. $\Delta t$) gives poor accuracy. The scheme is only first-order accurate in time.

- Crank-Nicolson (cf. Eq. 8.14 in M & V)

$$a_P T_P = a_E \frac{1}{2} T_E + a_W \frac{1}{2} T_W + \underbrace{a_E \frac{1}{2} T_E^o + a_W \frac{1}{2} T_W^o + \left(a_P^o - \frac{1}{2} a_E - \frac{1}{2} a_W\right) T_P^o}_{S_U}$$

$$a_E = \frac{k_e}{\delta x_e},\ a_W = \frac{k_w}{\delta x_w},\ a_P^o = \frac{\rho c_p \Delta x}{\Delta t}$$

$$a_P = \frac{1}{2}\left(a_E + a_W\right) + a_P^o$$

Also this scheme is implicit and unconditionally stable. In practice, however, it is less stable than the fully implicit scheme. Crank-Nicolson in time can be compared with central differencing in space, even though it is much more stable.

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start time marching
        start iteration
                solve t
        converged (implicit, C-N)? if no, next iteration
        last time step? if no, next time step
end of time marching
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Look at Examples 8.1 and 8.2