6.1 Introduction

So far we've studied the diffusion (heat conduction) equation

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+S=0,$$

and the convection–diffusion equation

$$\frac{d}{dx}\left(\rho c_p U T\right)=\frac{d}{dx}\left(k\frac{dT}{dx}\right)+S.$$

Now we will look at the Navier-Stokes (momentum) equations and the continuity equation. In 1D, and in unsteady form we have

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\frac{\partial\rho U}{\partial x}&=0\\ \frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}\left(\rho UU\right)&=-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial U}{\partial x}\right).\end{aligned} \tag{31}$$

We see that we have three unknowns and two equations; the third equation is the equation of state. Thus, from the continuity equation we get the density, from the momentum equation we get U, and the equation of state gives the pressure P. The above equation system is used for *compressible* flow, i.e. high-speed flow when the Mach number $Ma > 0.3$. Methods to solve this equation system will be treated in a later lecture.

If the fluid is (nearly) incompressible, the equation system above reads

$$\begin{aligned}\frac{\partial\rho U}{\partial x}&=0\\ \frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}\left(\rho UU\right)&=-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial U}{\partial x}\right),\end{aligned}$$

where we allow changes in ρ due to variation in temperature and concentration of species, but not due to pressure variations (i.e. $\partial\rho/\partial p = 0$). Now we have two equations and two unknowns. The problem is that we don't have any equation for pressure. Instead we have to use the continuity equation as an indirect equation for pressure.

Note that in the convective term above we have UU (a non-linear term); we'll treat one explicitly (i.e taking it from the old iteration level) and one implicitly (the one we solve for). This is quite similar to the convection-diffusion equation for temperature where we had UT in the convection term: U was supposed to be known, and we solved for T.

6.2 The staggered grid

Let's discretize the pressure gradient for the control volume below, assuming equidistant mesh ($\Delta x = \delta x_e = \delta x_w$)

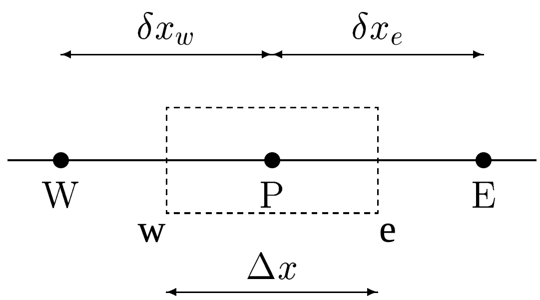
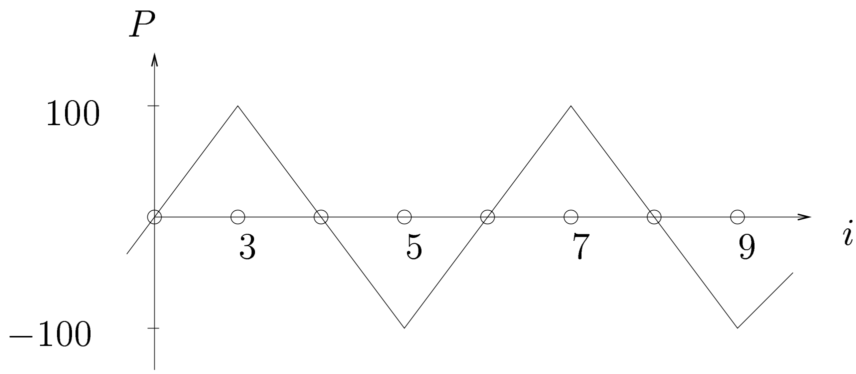


Figure. 1D control volume. Node P located in the middle of the control volume.

$$\int_w^e -\frac{\partial P}{\partial x} dx = \frac{P_W - P_E}{2}. \qquad (32)$$

We find that the pressure gradient at control volume P does not "feel" the pressure P_P at node P, i.e. the discretized pressure gradient is independent of P_P. This means that a highly oscillating pressure field as that below can occur.



We see that

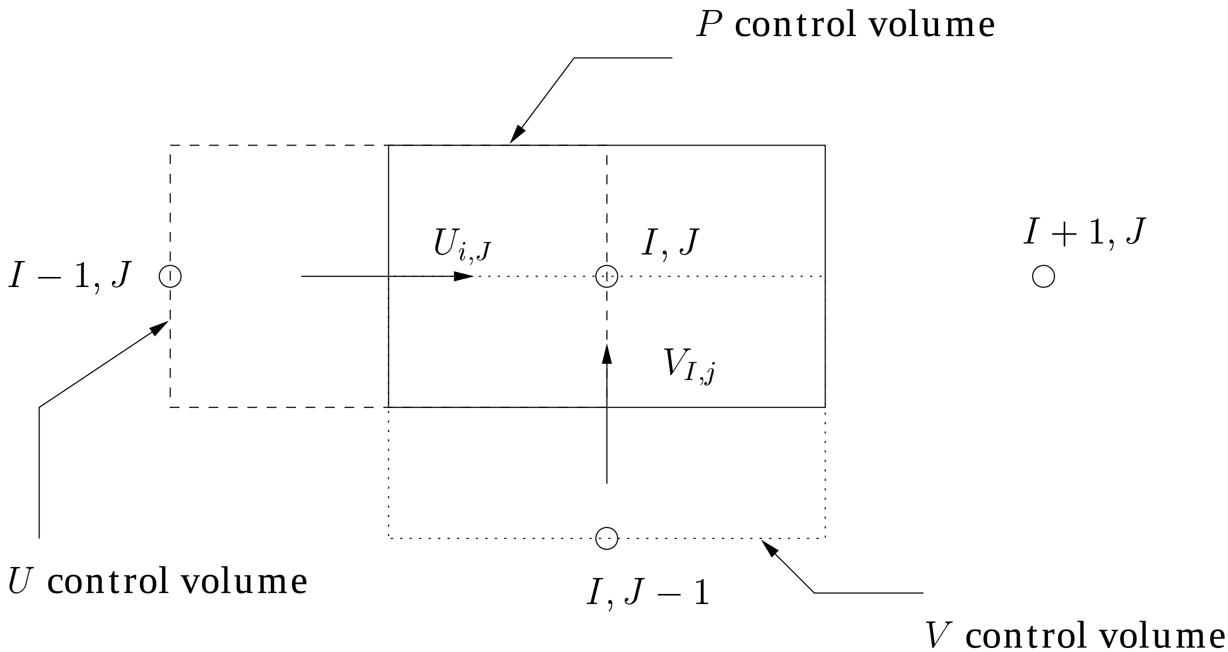
$$\int -\left(\frac{\partial P}{\partial x}\right)_4 dx = \frac{P_3 - P_5}{2} = \int -\left(\frac{\partial P}{\partial x}\right)_5 dx = \frac{P_4 - P_6}{2} dx =$$
$$\ldots = \int -\left(\frac{\partial P}{\partial x}\right)_8 dx = \frac{P_7 - P_9}{2} = 0$$

although P is highly oscillating. A pressure field like that in the figure above is called *checker-board* pressure field.

A remedy it to *stagger* the grid for the velocities.

- The U momentum equation includes $\partial P/\partial x$ and is thus staggered in the x direction;
- The V momentum equation includes $\partial P/\partial y$ and is thus staggered in the y direction;
- The W momentum equation includes $\partial P/\partial z$ and is thus staggered in the z direction,

see figure below.



Note that the pressure (main) control volume is denoted by upper-case indices (I, J), whereas a lower-case i-index means that the control volume has been staggered in the x direction (i, J), i.e. it is a U control volume. A lower-case j-index means that the control volume has been staggered in the y direction (I, j), i.e. it is a V control volume.

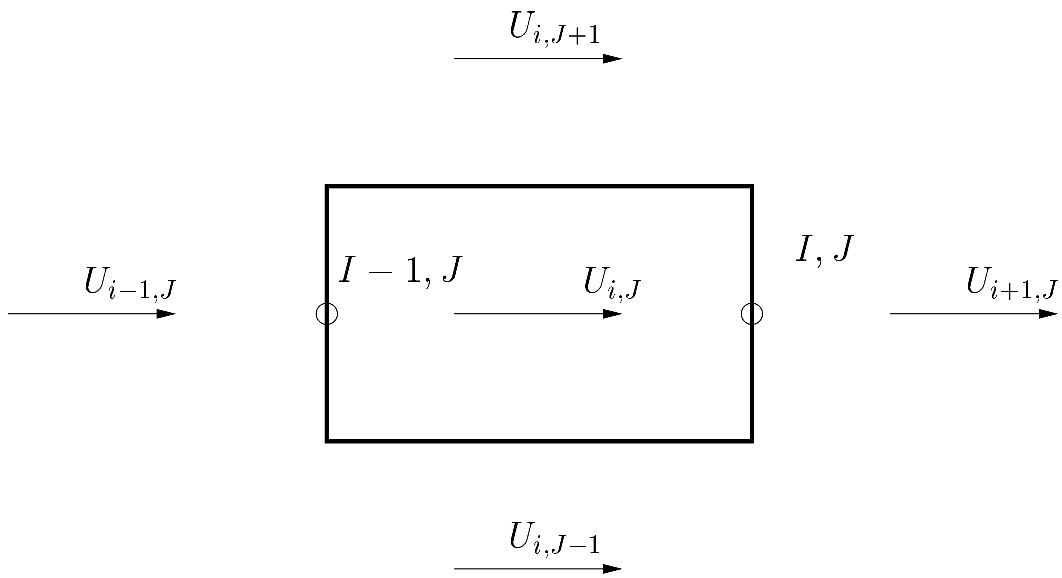
Let's discretize $\partial P/\partial x$ and $\partial P/\partial y$ over the U and V control volume, respectively; we obtain

$$
\begin{aligned}
\int_{\Delta V_{i,J}} -\left(\frac{\partial P}{\partial x}\right)_{i,J} dx &= P_{I-1,J} - P_{I,J} \\
\int_{\Delta V_{I,j}} -\left(\frac{\partial P}{\partial y}\right)_{I,j} dy &= P_{I,J-1} - P_{I,J},
\end{aligned}
\tag{33}
$$

see the figure above. As is seen, all pressure nodes are used, which prevents oscillating pressure.

6.3 The Momentum Equations

Discretize the 2D U equation over its control volume, see the figure below.



As mentioned at p. 41, the convective, non-linear term must be linearized so that one U is treated as known, and one is solved for; to differ between these two we denote the U which we solve for by Φ, so that the U momentum equation reads (see Eq. 31 and Eq. 6.1 in M & V)

$$\underbrace{\frac{\partial}{\partial x}(\rho U \Phi)}_{\text{Term I}} + \frac{\partial}{\partial y}(\rho V \Phi) = -\frac{\partial P}{\partial x} + \underbrace{\frac{\partial}{\partial x}\left(\mu \frac{\partial \Phi}{\partial x}\right)}_{\text{Term II}} + \frac{\partial}{\partial y}\left(\mu \frac{\partial \Phi}{\partial y}\right).$$

with $\Phi = U$. The diffusion and convection terms are discretized as in Chapter 4 and 5, except that now the control volume is staggered in the negative x direction. Term I is discretized as

$$\int_{\Delta V_{i,J}} \frac{\partial}{\partial x}(\rho U \Phi)\, dxdy = \int_{I-1,J}^{I,J} \int_{I,J-1/2}^{I,J+1/2} \frac{\partial}{\partial x}(\rho U \Phi)\, dxdy =$$
$$\int_{I,J-1/2}^{I,J+1/2} \left[(\rho U \Phi)_{I,J} - (\rho U \Phi)_{I-1,J}\right] dy = \left[(\rho U \Phi)_{I,J} - (\rho U \Phi)_{I-1,J}\right] \Delta y$$

A suitable discretization scheme has to be chosen for Φ at the faces [(I,J) and $(I-1,J)$] as in Chapter 5.

The diffusion term in the x direction (Term II) is discretized as

$$\int_{\Delta V_{i,J}} \frac{\partial}{\partial x}\left(\mu \frac{\partial \Phi}{\partial x}\right) dxdy = \int_{I-1,J}^{I,J} \int_{I,J-1/2}^{I,J+1/2} \frac{\partial}{\partial x}\left(\mu \frac{\partial \Phi}{\partial x}\right) dxdy =$$
$$\int_{I,J-1/2}^{I,J+1/2} \left[\left(\mu \frac{\partial \Phi}{\partial x}\right)_{I,J} - \left(\mu \frac{\partial \Phi}{\partial x}\right)_{I-1,J}\right] dy =$$
$$\left[\mu_{I,J} \frac{\Phi_{i+1,J} - \Phi_{i,J}}{\delta x_{I,J}} - \mu_{I-1,J} \frac{\Phi_{i,J} - \Phi_{i-1,J}}{\delta x_{I-1,J}}\right] \Delta y$$

The pressure term is discretized as in Eq. 33, i.e.

$$\int_{\Delta V_{i,J}} \frac{\partial P}{\partial x} dxdy = \int_{I-1,J}^{I,J} \int_{I,J-1/2}^{I,J+1/2} \frac{\partial P}{\partial x} dxdy =$$
$$\int_{I,J-1/2}^{I,J+1/2} \left(P_{I-1,J} - P_{I,J}\right) dy = \left(P_{I-1,J} - P_{I,J}\right) \Delta y$$

The discretized U momentum equation can now be written as (where we replace Φ by U)

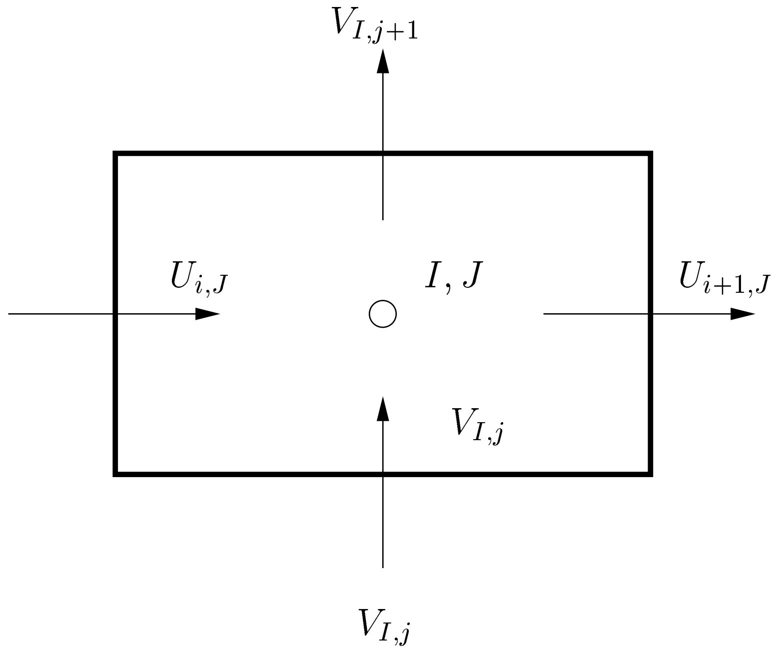
$$a_{i,J}U_{i,J} = \sum_{nb} a_{nb}U_{nb} + (P_{I-1,J} - P_{I,J})\,\Delta y$$
$$\sum_{nb} a_{nb}U_{nb} = a_W U_{i-1,J} + a_E U_{i+1,J} + a_S U_{i,J-1} + a_N U_{i,J+1}$$
$$a_{i,J} = a_W + a_E + a_S + a_N$$

6.4 The SIMPLE Algorithm

As mentioned in the beginning of this chapter, the continuity equation will be used as an indirect equation for pressure. First the momentum equations are solved, using an ”old” pressure (old values denoted by an asterix as superscript), to give U^* and V^* (see Eqs. 6.12-13 in M & V)

$$\begin{aligned} a_{i,J}U^*_{i,J} &= \sum_{nb} a_{nb}U^*_{nb} + \left(P^*_{I-1,J} - P^*_{I,J}\right)\Delta y \\ a_{I,j}V^*_{I,j} &= \sum_{nb} a_{nb}V^*_{nb} + \left(P^*_{I,J-1} - P^*_{I,J}\right)\Delta x. \end{aligned} \tag{34}$$

Note that U and V are not solved for the same control volumes, see figure on p. 43. Now we discretize the continuity equation over the main control volume (see figure below and compare with figure on p. 43)



$$\int_{\Delta V_{I,J}} \left[\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V)\right] dxdy =$$

$$\int_{I-1/2,J}^{I+1/2,J} \int_{I,J-1/2}^{I,J+1/2} \left[\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V)\right] dxdy =$$

$$\left[(\rho U)_{i+1,J} - (\rho U)_{i,J}\right] \Delta y + \left[(\rho V)_{I,j+1} - (\rho V)_{I,j}\right] \Delta x = 0.$$

Note that index $(I-1/2,J)$ is equal to (i,J), and index $(I+1/2,J)$ is equal to $(i+1,J)$, and so on. Introduce

$$U = U^* + U',\ V = V^* + V',\ P = P^* + P' \tag{35}$$

where U^* and V^* have been obtained from the momentum equations, and P^* was obtained at the previous iteration. Replace U and V in the continuity equation using Eq. 35 so that

$$\left\{[\rho(U^*+U')]_{i+1,J}-[\rho(U^*+U')]_{i,J}\right\}\Delta y +$$
$$\left\{[\rho(V^*+V')]_{I,j+1}-[\rho(V^*+V')]_{I,j}\right\}\Delta x = 0.$$

Rewriting this as an equation for U' and V' gives

$$\begin{aligned}
&\left[(\rho U')_{i+1,J}-(\rho U')_{i,J}\right]\Delta y +\\
&\left[(\rho V')_{I,j+1}-(\rho V')_{I,j}\right]\Delta x = b'\\
&b' = -\left[(\rho U^*\Delta y)_{i+1,J}-(\rho U^*\Delta y)_{i,J}\right.\\
&\left.+\,(\rho V^*\Delta x)_{I,j+1}-(\rho V^*\Delta x)_{I,j}\right]
\end{aligned} \tag{36}$$

where b' is the continuity error, and should, when convergence has been reached, be zero. We want to turn the equation above into an equation for P'. We use the momentum equations to obtain a relation between U', V' and P'. The U momentum equation (Eq. 34) is also valid for U' so that

$$a_{i,J}U'_{i,J}=\sum_{nb}a_{nb}U'_{nb}+\left(P'_{I-1,J}-P'_{I,J}\right)\Delta y. \tag{37}$$

We're interested in a direct relation between U' and P', and thus we rewrite the above equation as

$$\begin{aligned}
&a_{i,J}U'_{i,J}=\left(P'_{I-1,J}-P'_{I,J}\right)\Delta y\\
&\Rightarrow U'_{i,J}=d_{i,J}\left(P'_{I-1,J}-P'_{I,J}\right),\ d_{i,J}=\frac{\Delta y}{a_{i,J}}.
\end{aligned} \tag{38}$$

In the same way we obtain an relation between V' and P'

$$V'_{I,j}=d_{I,j}\left(P'_{I,J-1}-P'_{I,J}\right),\ d_{I,j}=\frac{\Delta x}{a_{I,j}}. \tag{39}$$

Insert Eqs. 38,39 into Eq. 36

$$\begin{aligned}
a_{I,J} P'_{I,J} &= \sum_{nb} a_{nb} P'_{nb} + b' \\
a_W &= (\rho d \Delta y)_{i,J}\,,\; a_E = (\rho d \Delta y)_{i+1,J} \\
a_S &= (\rho d \Delta x)_{I,j}\,,\; a_N = (\rho d \Delta x)_{I,j+1} \\
a_{I,J} &\equiv a_P = a_E + a_W + a_N + a_S \\
b' &= -\Big[(\rho U^* \Delta y)_{i+1,J} - (\rho U^* \Delta y)_{i,J} \\
&\quad + (\rho V^* \Delta x)_{I,j+1} - (\rho V^* \Delta x)_{I,j} \Big]
\end{aligned} \tag{40}$$

It should be stressed that the pressure-correction equation is a *correction* equation; the object is only to satisfy the continuity equation, i.e. to make the source term b' in Eq. 40 vanish. Thus, the fact that the pressure-correction equation was rewritten (from Eq. 37 to Eq. 38) does not influence the results at all.

The solution procedure for the equation system consisting of the continuity and the Navier-Stokes equation can be summarized as:

•1. Guess the pressure P^*

•2. Solve the Navier-Stokes equations (Eq. 34) $\to U^*$, V^*

•3. Solve the pressure-correction equation (Eq. 40) $\to P'$

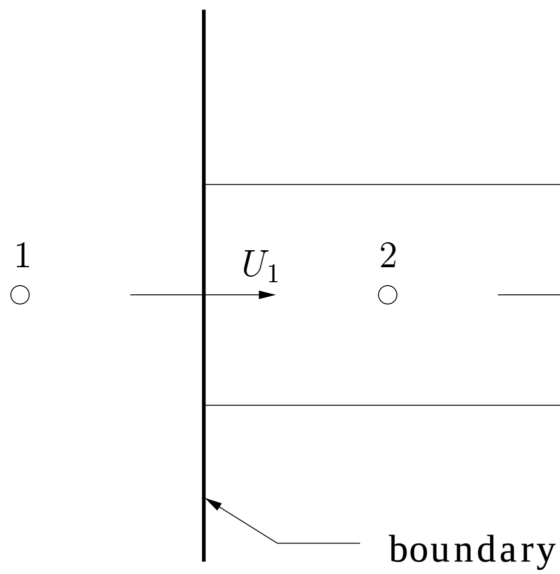
•4. Correct the velocities (see Eqs. 38, 39) and the pressure as

$$\begin{aligned}
U_{i,J} &= U^*_{i,J} + d_{i,J}\left(P'_{I-1,J} - P'_{I,J}\right) \\
V_{I,j} &= V^*_{I,j} + d_{I,j}\left(P'_{I,J-1} - P'_{I,J}\right) \\
P_{I,J} &= P^*_{I,J} + P'_{I,J}
\end{aligned} \tag{41}$$

•5. Repeat Step 2–4 till convergence.

Boundary Conditions

What are the boundary conditions for the P' equation? Usually the normal velocity component at a boundary is given (the exception is when a pressure boundary condition is used, see Section 9.5 in M & V). On a west (low x) boundary, for example,



Since U_1 is given, $U_1 = U_1^*$, i.e. U_1 should not be corrected. From Eq. 41 we see that this is satisfied if the coefficient between the near-boundary node and the boundary is set to zero; in this case $a_W = 0$. Thus the boundary conditions where the normal velocity is given is homogeneous Neumann boundary conditions, which we implement by setting $a_W = 0$ (west boundary), $a_E = 0$ (east boundary), and so on.

Near a west boundary, for example, the U momentum equations are solved for U_2, see the figure above. The pressure gradient in the U momentum equation is conveniently computed using the pressure at nodes 2 and 3. We find that we don't need any boundary condition for P. This is actually a result of that the velocity grid does not fully cover the computational domain, but stops half a control volume inside the physical boundary.

An Example

Consider a one-dimensional configuration with only two main control volumes, see figure below. The discretized pressure correction equation in one dimension can be written as

$$\begin{aligned} a_P P'_P &= a_E P'_E + a_W P'_W + b \\ a_E &= (\rho d)_{i+1},\ a_W = (\rho d)_i,\ a_P = a_E + a_W \\ b &= (\rho U^*)_i - (\rho U^*)_{i+1} \equiv \dot{m}^*_i - \dot{m}^*_{i+1} \end{aligned} \tag{42}$$

Assume constant density $\rho = 1$, viscosity $\mu = 0.1$ and $\Delta x = 1$. The boundary conditions for the velocities are $U_2 = U_4 = 1$ (note that U_1 is never used since the U grid is staggered half a control volume to the left). We solve the one-dimensional U momentum equation using upwind differencing

$$\begin{aligned} a^U_i U^*_i &= a^U_E U^*_{i+1} + a^U_W U^*_{i-1} + P^*_W - P^*_P + b \\ a^U_i &= a^U_E + a^U_W \end{aligned} \tag{43}$$

We solve the equation for face 3. From the initial, guessed

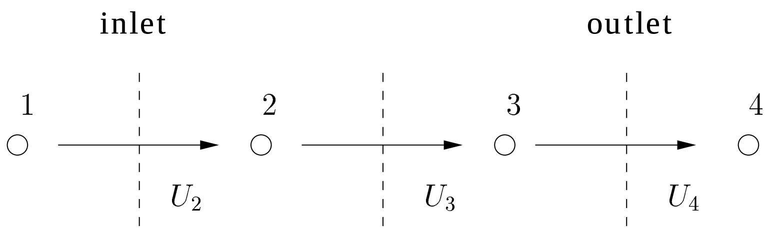


Figure 1: A 1D grid for velocity U and pressure P.

conditions $U_3^* = P_2^* = P_3^* = 0$ we have

$$a^U_{W,3} = \frac{\mu}{\Delta x} + \max\left\{\frac{1}{2}(U_2^* + U_3^*), 0\right\} = 0.6$$
$$a^U_{E,3} = \frac{\mu}{\Delta x} + \max\left\{0, -\frac{1}{2}(U_2^* + U_3^*)\right\} = 0.1$$
$$a^U_3 = a^U_{E,3} + a^U_{W,3} = 0.7,\ b_3 = -0.05$$

where b_3 is a retarding force ($= 0.05$ per unit length) due to wall-friction. We obtain $U_3^* = (0.6 \cdot 1 + 0.1 \cdot 1 - 0.05)/0.7 = 0.929$.

Now the P'_P equation will be solved.

CELL 2

We have

$$a_{W,2} = 0,\ d_{i,3} = \frac{1}{a^U_3} = 1.43\ \Rightarrow a_{E,2} = 1.43,$$

and from Eq. 42 we get $a_{P,2} = 1.43$, $b_2 = 0.071$. The discretized equation for the P'_P equation for cell 2 thus reads

$$1.43P'_2 = 1.43P'_3 + 0.071 \tag{44}$$

CELL 3

For cell 3 we have $a_{E,3} = 0$. The west coefficient is the same as the east coefficient for cell 2, i.e. $a_{W,3} = a_{E,2}$, which gives $a_{P,3} = 1.43$. The source term $b_3 = -0.071$, so the discretized equation for node 3 can be written

$$1.43P'_3 = 1.43P'_2 - 0.071 \tag{45}$$

We see that the equation system formed by Eqs. 44 and 45 is singular (the determinant is zero). This is always the case for the pressure correction equation because we have zero normal gradient $\partial P'/\partial n = 0$ ($dP'/dx = 0$ in this example) at all boundaries, which means that P' is determined up to an arbitrary additive constant. To get around this problem we fix this constant by setting P' to zero at a chosen cell (cell 2, for instance), and omit this cell's equation from the equation system. Here we set $P'_2 = 0$ and we get from Eq. 45 $P'_3 = -0.05$. The pressure is corrected according to

$$P_P = P^*_P + P'_P, \tag{46}$$

which gives $P_3 = -0.05$, and the U^*_3 velocity is corrected as

$$U_3 = U^*_3 + d_2(P'_2 - P'_3) = 0.929 + 1.43\,(0 + 0.05) = 1.00 \tag{47}$$

Thus we see that the pressure correction equation corrects the velocities so that the continuity is satisfied. In this example, it happens that the momentum equation is also satisfied, i.e. if the U_3 momentum equation is solved once again with new pressure (P_3) and new velocity (U_3), we get $U_3 = 1.00$. Thus the momentum and the continuity equations are satisfied.

It was mentioned above that the equation system for the pressure correction equation was singular. Note that this is only a problem for *direct* solvers (i.e. when we invert the matrix); when we use iterative solvers such as Gauss-Seidel, we solve the pressure correction equations without worrying about that the system is singular.

Under-relaxation

Since the momentum equations are non-linear, we must use under-relaxation. This means that when we obtain U from the solver (for example, Gauss-Seidel or TDMA) we compute the new U value as a blend of the value from the solver (U^{solver}) and the old value U^*, i.e.

$$U_{i,J} = U^*_{i,J} + \alpha(U^{solver}_{i,J} - U^*_{i,J}) \tag{48}$$

where $0 \leq \alpha \leq 1$. When we're using small α's, the changes between successive iterations is slowed down ($\alpha = 0$ corresponds to no change at all), and $\alpha = 1$ corresponds to no under-relaxation. The U momentum equation can be written (see Eq. 34)

$$a_{i,J}U^{solver}_{i,J} = \sum_{nb} a_{nb}U_{nb} + S_U$$

with the pressure gradient included in the source term S_U. Insert this expression into Eq. 48 gives

$$U_{i,J} = U^*_{i,J} + \alpha\left[\frac{1}{a_{i,J}}\left(\sum_{nb} a_{nb}U_{nb} + S_U\right) - U^*_{i,J}\right],$$

which can be rewritten as

$$\frac{a_{i,J}}{\alpha}U_{i,J} = \frac{a_{i,J}}{\alpha}U^*_{i,J} + \sum_{nb} a_{nb}U_{nb} + S_U - a_{i,J}U^*_{i,J}.$$

Rearranging gives

$$a_{i,J}^{mod} U_{i,J} = \sum_{nb} a_{nb} U_{nb} + S_U^{mod}$$
$$a_{i,J}^{mod} = \frac{a_{i,J}}{\alpha},\ S_U^{mod} = S_U + a_{i,J}^{mod}(1-\alpha)U_{i,J}^*.$$

We find that introducing under-relaxation directly into the equation system is conveniently carried out by modifying the diagonal coefficient $a_{i,J}$ and the source term S_U.