Collocated grids

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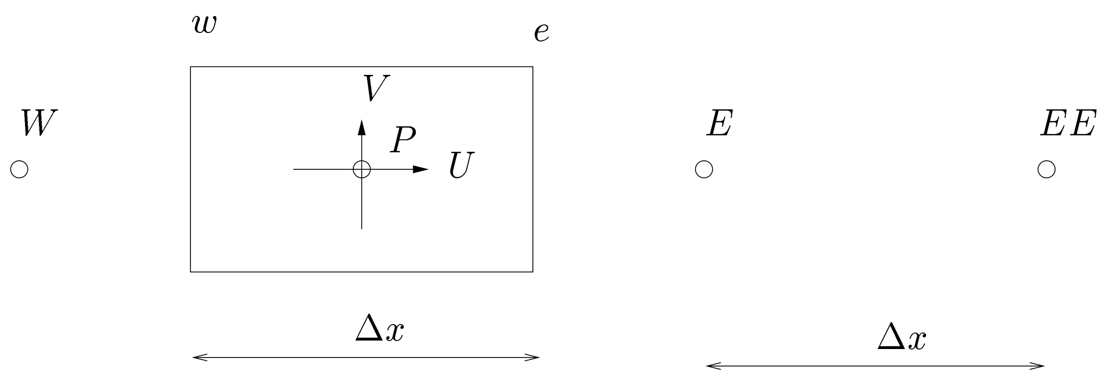
In 1983 Rhie & Chow [6] proposed a *non-staggered* grid arrangement for velocity components, see the figure below. For details, see e.g. Refs. [2, 1].



The discretized U momentum equation in 1D can be written

$$a_P U_P = a_E U_E + a_W U_W - \frac{dP}{dx}\delta V, \tag{49}$$

and the pressure correction equation has the form

$$\begin{aligned} a_P P'_P &= a_E P'_E + a_W P'_W + b \\ a_E &= (\rho d)_e,\ a_W = (\rho d)_w,\ a_P = a_E + a_W \\ b &= (\rho U)_w - (\rho U)_e \end{aligned} \tag{50}$$

Now we want to compute the face velocities of U for computing the source term b (=continuity equation). Pure central differencing would give (constant density; equidistant mesh, see the figure above)

$$\frac{1}{2}(U_E + U_P) - \frac{1}{2}(U_P + U_W) = 0 \ \Rightarrow U_E - U_W = 0$$

In this way we would face the same problems as when we tried to discretize the pressure gradient in Chapter 6 (see

Eq. 32 on p. 41), i.e. the velocity at node P is not used and we would admit oscillating velocity and pressure fields. Instead we interpolate the velocity U_P and the pressure gradient separately. First we solve the U momentum equation (Eq. 49) as usual. Then we create a velocity $U^\diamond$ in which the pressure gradient is excluded, i.e. (see Eq. 49)

$$U_P^\diamond = U_P + \left(\frac{\delta V}{a_P}\frac{dP}{dx}\right)_P. \tag{51}$$

Linear intepolation is used to obtain the $U^\diamond$ field at the faces, i.e.

$$U_e^\diamond = \frac{1}{2}\left(U_E^\diamond + U_P^\diamond\right). \tag{52}$$

To find the U field at the faces we add the pressure gradient to $U^\diamond$. The important point is that we don't interpolated dP/dx from the nodes, but we *evaluate* dP/dx at the faces, i.e.

$$U_e = U_e^\diamond - \left(\frac{\delta V}{a_P}\frac{dP}{dx}\right)_e \Rightarrow U_e = U_e^\diamond - \left(\frac{\delta V}{a_P}\right)_e \frac{P_E - P_P}{\Delta x}. \tag{53}$$

The procedure given above to obtain the face velocite U_e is called Rhie-Chow interpolation after its inventors. Evaluating this expression by inserting Eqs. 51 and 52 into Eq. 53, assuming constant a_P and density (δV is constant since the mesh spacing is constant) gives

$$\begin{aligned}
U_e &= \frac{1}{2}\left(U_E + U_P\right) + \frac{\delta V}{2a_P}\left[\left(\frac{dP}{dx}\right)_E + \left(\frac{dP}{dx}\right)_P\right] - \frac{\delta V}{a_P}\frac{P_E - P_P}{\Delta x} \\
&= \frac{1}{2}\left(U_E + U_P\right) + \frac{\delta V}{2a_P}\left[\frac{P_{EE} - P_P}{2\Delta x} + \frac{P_E - P_W}{2\Delta x}\right] - \frac{\delta V}{a_P}\frac{P_E - P_P}{\Delta x} \\
&= \frac{1}{2}\left(U_E + U_P\right) + \frac{\delta V}{4a_P\Delta x}\left[P_{EE} - 3P_E + 3P_P - P_W\right].
\end{aligned}$$

We find that the Rhie-Chow interpolation is the same as adding a pressure term, which is proportional to a third derivative of the pressue d^3P/dx^3. The object of this term is to eliminate pressure oscillations, and thereby also velocity oscillations. In the continuity equation (Eq. 54) the term is proportional to a fourth-order derivative term d^4P/dx^4. This is shown by inserting the expression for U_e and a corresponding expression for U_w into the continuity equation, i.e.

$$\begin{aligned}\int_w^e \frac{dU}{dx}dx = U_e - U_w = \frac{1}{2}U_E + C_1(P_{EE} - 3P_E + 3P_P - P_W)\\ -[\frac{1}{2}U_W + C_1(P_E - 3P_P + 3P_W - P_{WW})]\\ = \frac{1}{2}(U_E - U_W) + C_1(P_{EE} - 4P_E + 6P_P - 4P_W + P_{WW})\\ C_1 = \frac{\delta V}{4a_P\Delta x}\end{aligned} \tag{54}$$

(note that the equation above is the source term b in Eq. 50) The fourth derivative stencil in pressure can be shown to be a dissipative term, i.e. a damping term (see *further reading* at the www-page).

An Example

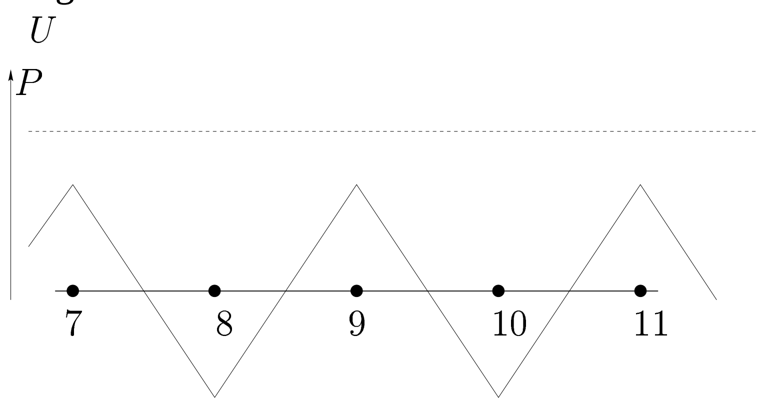
In this example we will show that without the Rhie-Chow interpolation, the discretized equations support an oscillating pressure field. Let's take a simple 1D flow for which the discretized U equation can be written as

$$\begin{aligned} a_P^U U_P^* &= a_E^U U_E^* + a_W^U U_W^* + P_W^* - P_E^* \\ a_P^U &= a_E^U + a_W^U \end{aligned} \tag{55}$$

and the P' equation as

$$\begin{aligned} a_P P'_P &= a_E P'_E + a_W P'_W + b \\ a_E &= (\rho d)_e,\ a_W = (\rho d)_w,\ a_P = a_E + a_W \\ b &= (\rho U^*)_w - (\rho U^*)_e \equiv \dot{m}_w^* - \dot{m}_e^* \end{aligned} \tag{56}$$

Assume that at a certain iteration we have a velocity field (dashed line) and pressure field (solid line) as show in the figure below.



The U velocity is constant and equal to one, and the pressure oscillates between ± 0.5. We take a closer look at the discretized equations for node 9 in the figure above. We assign some reasonable values on the coefficients in Eq. 55 (see the example at p. 51)

$$\begin{aligned} a_{E,9} &= a_{W,9} = 1.4 \\ a_{P,9} &= 2.8;\ b_9 = 0 \end{aligned} \tag{57}$$

The source term (the continuity equation) $b_9 = 0$ because the velocity $U = 1$ at all nodes. Thus the solution to the P' equation is $P'_9 = 0$ (actually $P'_i = 0$ for all nodes since $U_i = 1$ for all nodes). Thus the pressure will not be changed and $P_i = P^*_i = \pm 0.5$

Now we turn to the U momentum equation. Here we set (see the example at p. 51)

$$\begin{aligned} &a^U_{E,9} = 0.1;\ a^U_{W,9} = 0.6 \\ &a^U_{P,9} = 0.7 \end{aligned} \tag{58}$$

With pressure field as in the figure above, the pressure gradient term $-(P_E - P_W)$ in Eq. 55 is zero. Thus, we can see that with $U = 1$ and an oscillating pressure field both the continuity equation and the U momentum equation are satisfied. This is clearly an unphysical solution.

Now let's see what happens if we use Rhie-Chow interpolation. The only change is that the source term b in the pressure correction equation is computed as Eq. 54, i.e.

$$\begin{aligned} b_9 &= \frac{1}{2}(U_{10} - U_8) + C_1(P_{11} - 4P_{10} + 6P_9 - 4P_8 + P_7) \\ C_1 &= \frac{\delta V}{4a^U_P \Delta x} = \frac{1}{4 \cdot 0.7 \cdot 1} = 0.36 \end{aligned} \tag{59}$$

Inserted values of U and P gives $b_9 = 0.36 \cdot 8 = 2.86$. This means that the residual for the continuity equation is not zero. It will result in a P' field which is not zero. The pressure field and the velocity field will be corrected. The iteration process will go on and the final solution will be $U = 1$ and $P = const$, which is the correct solution.