Reynolds Stress Models

Deriving the $\overline{u_i u_j}$ equation

Set up the momentum equation for the instantaneous velocity $U_i = \bar{U}_i + u_i \rightarrow$ Eq. (1)

Time average $\rightarrow$ Eq. (2)

Subtract Eq. (2) from Eq. (1) $\rightarrow$ Eq. (3)

Do the same procedure for $U_j \rightarrow$ Eq. (4)

Multiply Eq. (3) with u_j and Eq. (4) with u_i, time average and add them together $\rightarrow$ Eq. for $\overline{u_i u_j}$

The $\overline{u_i u_j}$-equation (Reynolds Stress equation) has the form:

$$
\begin{aligned}
\underbrace{\bar{U}_k \frac{\partial \overline{u_i u_j}}{\partial x_k}}_{C_{ij}} = & \underbrace{-\overline{u_i u_k}\frac{\partial \bar{U}_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial \bar{U}_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{\Phi_{ij}} \\
& \underbrace{-\frac{\partial}{\partial x_k}\left[\overline{u_i u_j u_k} + \frac{\overline{p u_j}}{\rho}\delta_{ik} + +\frac{\overline{p u_i}}{\rho}\delta_{jk} - \nu \frac{\partial \overline{u_i u_j}}{\partial x_k}\right]}_{D_{ij}} \\
& \underbrace{- 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}}_{\varepsilon_{ij}}
\end{aligned}
$$

which symbolically can be written:

$$
C_{ij} - D_{ij} = P_{ij} + \Phi_{ij} - \varepsilon_{ij}
$$

The k equation

The turbulent kinetic energy is the sum of all normal Reynolds stresses, i.e.

$$k = \frac{1}{2}\left(\overline{u^2} + \overline{v^2} + \overline{w^2}\right) \equiv \frac{1}{2}\overline{u_i u_i}$$

By taking the trace (setting indices $i = j$) in the equation for $\overline{u_i u_j}$ we get the equation for the turbulent kinetic energy equation:

$$\underbrace{\bar{U}_j \frac{\partial k}{\partial x_j}}_{C_k} = -\underbrace{\overline{u_i u_j}\frac{\partial \bar{U}_i}{\partial x_j}}_{P_k} \underbrace{-\frac{\partial}{\partial x_j}\left\{\overline{u_j\left(\frac{p}{\rho} + \frac{1}{2}u_i u_i\right)} - \nu \frac{\partial k}{\partial x_j}\right\}}_{D_k}$$
$$- \underbrace{\nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}}_{\varepsilon}$$

which symbolically can be written:

$$C_k - D_k = P_k - \varepsilon$$

Modelling assumptions

Now we'll model the unknown terms in the $\overline{u_i u_j}$ equation. This will give us the Reynolds Stress Model (RSM) where a (modelled) transport equation is solved for each stress. Later on, we will introduced a simplified algebraic model, which is called the Algebraic Stress Model (ASM)

Physical meaning:

– P_{ij}, P_k are production terms of $\overline{u_i u_j}$ and k

- Φ_{ij} is the pressure-strain correlation term, which promotes isotropy of the turbulence
- ε, ε_{ij} are dissipation (i.e. transformation of mechanical energy into heat in the small-scale turbulence) of k and $\overline{u_i u_j}$, respectively.

Production term, RSM, ASM:

$$P_{ij} = -\rho\overline{u_i u_k}\frac{\partial \bar{U}_j}{\partial x_k} - \rho\overline{u_j u_k}\frac{\partial \bar{U}_i}{\partial x_k}$$

Production term, $k - \varepsilon$:

$$-\overline{u_i u_j} = \nu_t\left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}k$$

$$P_k = \mu_t\left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right)\frac{\partial \bar{U}_i}{\partial x_j}$$

Diffusion term in the k & ε-equations, RSM, ASM:

$$D_k = \frac{\partial}{\partial x_j}\left[\left(\nu + c_k\,\overline{u_j u_m}\,\frac{k}{\varepsilon}\right)\frac{\partial k}{\partial x_m}\right] \tag{85}$$

Diffusion term in the k & ε-equations, $k - \varepsilon$:

$$D_k = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

Dissipation term:

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij}$$

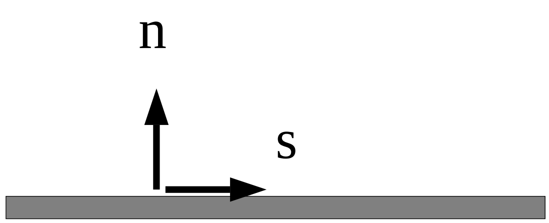
Pressure-Strain Redistribution term:

$$\Phi_{ij} = \Phi_{ij,1} + \Phi_{ij,2} + \Phi'_{ij,1} + \Phi'_{ij,2} \tag{86}$$

where

$$\Phi_{ij,1} = -c_1\frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}\delta_{ij}k\right),\ \Phi_{ij,2} = -c_2\left(P_{ij} - \frac{2}{3}\delta_{ij}P\right) \tag{87}$$

<u>Wall correction</u>



Wall-corrections for $\overline{u_n^2}$:

$$\Phi'_{nn,1} = -2c'_1\frac{\varepsilon}{k}\overline{u_n^2}f,\ f = \frac{k^{\frac{3}{2}}}{2.55x_n\varepsilon}$$

Wall-corrections for $\overline{u_s^2}$:

$$\Phi'_{ss,1} = c'_1\frac{\varepsilon}{k}\overline{u_n^2}f$$

Wall-corrections for $\overline{u_s u_n}$:

$$\Phi'_{sn,1} = -\frac{3}{2}c'_1\frac{\varepsilon}{k}\overline{u_s u_n}f$$

The modeled $\overline{u_i u_j}$ equation

The models for diffusion, pressure-strain and dissipation (see Eqs. 85,86,87 and page 103) gives

$$
\begin{aligned}
\bar{U}_k \frac{\partial \overline{u_i u_j}}{\partial x_k} = & \quad \text{convection} \\
-\overline{u_i u_k}\frac{\partial \bar{U}_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial \bar{U}_i}{\partial x_k} & \quad \text{production} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}\delta_{ij} k\right)\Phi_{ij,1} & \quad \text{(slow part)} \\
-c_2\left(P_{ij} - \frac{2}{3}\delta_{ij} P\right)\Phi_{ij,2} & \quad \text{(rapid part)} \\
+c_1' \rho \frac{\varepsilon}{k}\ [\ \overline{u_k u_m} n_k n_m \delta_{ij} - \frac{3}{2}\overline{u_i u_k} n_k n_j & \\
-\frac{3}{2}\overline{u_j u_k} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right]\Phi'_{ij,1} & \quad \text{(wall, slow part)} \\
+c_2'\ [\ \phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\phi_{ik,2} n_k n_j & \\
-\frac{3}{2}\phi_{jk,2} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right]\Phi'_{ij,2} & \quad \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} & \quad \text{viscous diffusion} \\
+\frac{\partial}{\partial x_k}\left[\left(\nu + c_k\ \overline{u_k u_m}\ \frac{k}{\varepsilon}\right)\frac{\partial \overline{u_i u_j}}{\partial x_m}\right] & \quad \text{turbulent diffusion} \\
-\frac{2}{3}\varepsilon \delta_{ij} & \quad \text{dissipation}
\end{aligned}
$$

ASM

Algebraic Reynolds Stress Model is a simplified Reynolds Stress Model

The RSM and $k-\varepsilon$ models are written in symbolic form (see pages 100 & 101) as:

$$\text{RSM} : C_{ij} - D_{ij} = P_{ij} + \Phi_{ij} - \varepsilon_{ij}$$

$$k-\varepsilon : \; C_k - D_k = P_k - \varepsilon$$

The assumption in ASM is that the transport (convective and diffusive) of $\overline{u_i u_j}$ is related to that of k, i.e.

$$C_{ij} - D_{ij} = \frac{\overline{u_i u_j}}{k} (C_k - D_k)$$

which gives:

$$\overline{u_i u_j} = \frac{2}{3}\delta_{ij} k + \frac{k}{\varepsilon} \frac{(1-c_2)\left(P_{ij} - \frac{2}{3}\delta_{ij} P\right) + \Phi'_{ij,1} + \Phi'_{ij,2}}{c_1 + P/\varepsilon - 1}$$

RSM versus ASM

Their ability to model turbulence is for many flows is very similar.

ASM has (had) an reputation of being simple and easy to implement: true for boundary layer flow where

$$-\overline{uv} = \underbrace{\frac{2}{3}(1-c_2)\frac{c_1 - 1 + c_2 P_k/\varepsilon}{(c_1 - 1 + p_k/\varepsilon)}}_{c_\mu} \frac{k^2}{\varepsilon} \frac{\partial \bar{U}}{\partial y}$$

For elliptic, recirculating flow, ASM is fairly unstable.

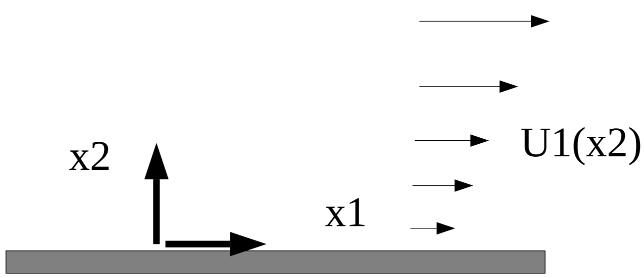
As a consequence an implementation of ASM is more difficult than of RSM.

Explicit ASM

Pope (1975) managed to derive an explicit expression for ASM in 2D (at page 105 it is implicit): Later this was extended to 3D by Gatski & Speziale (1993)

This new explicit ASM is considerable more stable from a numerical point of view than the old implicit ASM

Simple shear flow



Let us study simple shear flow where $\bar{U}_2 = 0,\ \bar{U}_1 = \bar{U}_1(x_2)$

In general the production P_{ij} has the form (see page 100):

$$P_{ij} = -\overline{u_i u_k}\frac{\partial \bar{U}_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial \bar{U}_i}{\partial x_k}$$

In this special case we get:

$$P_{11} = -2\overline{u_1 u_2}\frac{\partial \bar{U}_1}{\partial x_2},\ P_{12} = -\overline{u_2^2}\frac{\partial \bar{U}_1}{\partial x_2},\ P_{22} = 0$$

Is $\overline{u_2^2}$ zero because its production term P_{22} is zero?

The sympathetic term Φ_{ij} which takes from the rich (i.e. $\overline{u_1^2}$) and gives to the poor (i.e. $\overline{u_2^2}$) saves the unfair situation! $\Phi_{ij,1}$ has the form (see page 103):

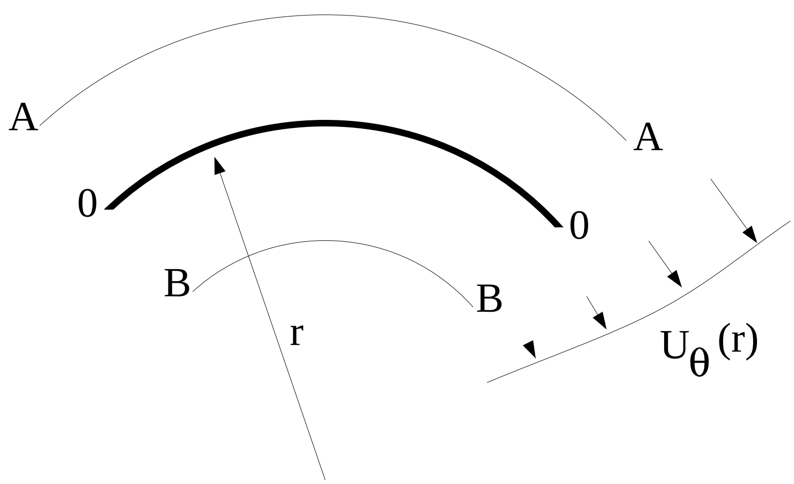
$$\Phi_{ij,1} = -c_1 \frac{\varepsilon}{k} \left(\overline{u_i u_j} - \frac{2}{3} k \delta_{ij} \right)$$

and we get:

$$\Phi_{22,1} = c_1 \frac{\varepsilon}{k} \left(\frac{2}{3} k - \overline{u_2^2} \right)$$

Note also that the dissipation term for the $\overline{u_1 u_2}$ is zero, but it takes the value $\frac{2}{3}\varepsilon$ for the $\overline{u_1^2}$ and $\overline{u_2^2}$ equations (see page 100)

Curvature effects



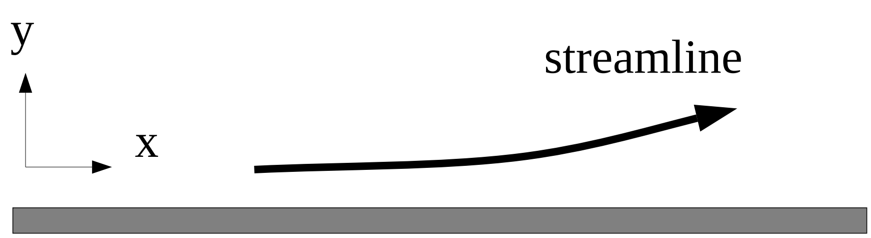
A polar coordinate system $r - \theta$ with $\hat{\theta}$ locally aligned with the streamline is introduced. The radial momentum equation degenerates to

$$\frac{\rho U_\theta^2}{r} - \frac{\partial p}{\partial r} = 0 \tag{88}$$

If the fluid is displaced by some disturbance (e.g. turbulent fluctuation) outwards to level A, it encounters a pressure gradient larger than at $r = r_0$, as $(U_\theta)_A > (U_\theta)_0$, which from Eq.(1) gives $(\partial p/\partial r)_A > (\partial p/\partial r)_0$. Hence the fluid is forced back to $r = r_0$.

Streamlines are often curved (see figure below) either due to flow phenomena (e.g. separation) or due to curved boundaries (e.g. airfoils)

The turbulence is strongly affected by curvature; Reynolds stress models (ASM/RSM) respond correctly to streamline curvature, whereas eddy viscosity models such as $k-\varepsilon$ don't



Weak Curvature: $\partial \bar{V}/\partial x \simeq 0.01 \times \partial \bar{U}/\partial y$, $\overline{u^2} \simeq 5\overline{v^2}$

The production terms due to rotational strains ($\partial \bar{U}/\partial y,\ \partial \bar{V}/\partial x$)

for ASM/RSM (see page 100) are:

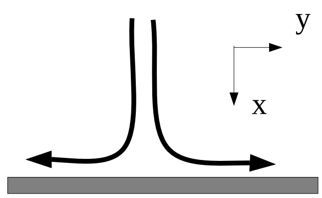
$$\text{RSM, } \overline{u^2} - \text{eq. :} \quad P_{11} = -2\overline{uv}\frac{\partial U}{\partial y}$$

$$\text{RSM, } \overline{uv} - \text{eq. :} \quad P_{12} = -\overline{u^2}\frac{\partial V}{\partial x} - \overline{v^2}\frac{\partial U}{\partial y}$$

$$\text{RSM, } \overline{v^2} - \text{eq. :} \quad P_{22} = -2\overline{uv}\frac{\partial V}{\partial x}$$

$$k-\varepsilon \quad P_k = \nu_t\left(\frac{\partial \bar{U}}{\partial y} + \frac{\partial \bar{V}}{\partial x}\right)^2$$

Stagnation flow



The $k-\varepsilon$ model does not model the normal stresses properly,whereas ASM/RSM do. The production for RSM/ASM and $k-\varepsilon$ model due to $\partial\bar{U}/dx$ and $\partial\bar{V}/dy$ is:

$$k-\varepsilon : P_k = 2\nu_t\left\{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right\}$$

$$RSM : 0.5\left(P_{11} + P_{22}\right) = -\overline{u^2}\frac{\partial U}{\partial x} - \overline{v^2}\frac{\partial V}{\partial y}$$

Since $\overline{u^2}$ and $\overline{v^2}$ will not differ drastically and since $\partial\bar{U}/dx = -\partial\bar{V}/dy$ due to continuity the production term with RSM/ASM will be zero; with $k-\varepsilon$, however, the production will be large!

RSM/ASM versus $k-\varepsilon$ models

- Advantages with $k-\varepsilon$ models (or eddy viscosity models):

  i) simple due to the use of an isotropic eddy (turbulent) viscosity

  ii) stable via stability-promoting second-order gradients in the mean-flow equations

  iii) work reasonably well for a large number of engineering flows

- Disadvantages:

  i) isotropic, and thus not good in predicting normal stresses ($\overline{u^2}, \overline{v^2}, \overline{w^2}$)

  ii) as a consequence of i) it is unable to account for curvature effects

  iii) as a consequence of i) it is unable to account for irrotational strains

- Advantages with ASM/RSM:

  i) the production terms need not to be modelled

ii) thanks to i) it can selectively augment or damp the stresses due to curvature effects, buoyancy etc.

- Disadvantages with ASM/RSM:

i) complex and difficult to implement, especially ASM

ii) numerically unstable because small stabilizing second-order derivatives in the momentum equations (only laminar diffusion)

iii) CPU consuming

Conclusions

Reynolds stress models can model many flows where simple $k - \varepsilon$ models fail; examples are:

i) flows where streamline curvature – or curvature of solid boundaries – is important

ii) flows affected of buoyancy

iii) flow near stagnation points

iv) rotating flows