Two Equation Models

The $k-\varepsilon$ Model

The $k-\varepsilon$ is an eddy-viscosity model. The dimensions of the laminar (kinematic) viscosity $\nu = \mu/\rho$ is $[\mathrm{m}^2/\mathrm{s}]$, which also must be the dimensions for the turbulent viscosity. Thus, we can write

$$\nu_t \propto \mathcal{U}\ell \; [\mathrm{m}^2/\mathrm{s}]$$

where $\mathcal{U}$ and ℓ denote a turbulent velocity and length scale, respectively. The turbulent length scale is taken as the square-root of the turbulent kinetic energy $k\,[\mathrm{m}^2/\mathrm{s}^2]$, i.e.

$$\nu_t \propto \sqrt{k}\ell$$

For finding k a transport equation will be used. We could also use a transport equation for the turbulent length scale ℓ, but it has been found that it is better to solve an equation for the dissipation ε of k. Thus we want to express ν_t in k and ε, i.e.

$$\nu_t = c_\mu k^a \varepsilon^b$$

where c_μ is a constant to be determined, and where a and b will be determined by dimensional analysis. The dimension of ε is $[\mathrm{m}^2/\mathrm{s}^3]$. The dimensions for the above equation must be the same on the left-hand and the right-hand side, and this will give us a and b. The dimensions are:

$$\frac{\mathrm{m}^2}{\mathrm{s}} = \left(\frac{\mathrm{m}^2}{\mathrm{s}^2}\right)^{\mathrm{a}} \left(\frac{\mathrm{m}^2}{\mathrm{s}^3}\right)^{\mathrm{b}}.$$

This gives us one equation for meter ([m]) and one for seconds ([s]), i.e.

$$\begin{aligned} \mathrm{m}: \quad 2 &= 2a + 2b \\ \mathrm{s}: \quad -1 &= -2a - 3b, \end{aligned}$$

which has the solution $a = 2$, $b = -1$. We get the following expression for ν_t

$$\nu_t = c_\mu \frac{k^2}{\varepsilon}, \; \nu_{tot} = \nu + \nu_t \tag{70}$$

The turbulent kinetic energy k and its dissipation ε are determined from their transport equations.

From the Navier-Stokes equations we can derive an exact equation for k (below it is given in boundary-layer form):

$$\frac{\partial \bar{U} k}{\partial x} + \frac{\partial \bar{V} k}{\partial y} = -\overline{uv}\frac{\partial \bar{U}}{\partial y} - \frac{\partial}{\partial y}\left[\frac{1}{\rho}\overline{pv} + \overline{vk'} - \nu\frac{\partial k}{\partial y}\right] - \nu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}.$$

On the left-hand side: convection. On the right-hand side: production, diffusion and dissipation. The diffusion has three parts: due to pressure, velocity fluctuations and viscosity.

We see that the above equation includes unknown terms such as $\overline{uv}$, $\overline{pv}$ and $\overline{\rho v k'}$; the last term in the k equation represents dissipation and it is obtained from its own transport equation. The unknown terms must be *modelled*.

• Production term $P_k = -\rho\overline{uv}\partial\bar{U}/\partial y$: the shear stress is modelled in the same way as for the momentum equations, i.e. using Boussinesq assumption so that

$$P_k = \nu_t \left(\frac{\partial \bar{U}}{\partial y}\right)^2$$

• Diffusion term: for the velocity correlations a gradient hypothesis, i.e. we assume that the term transports k from regions of high k to regions of low k (cf. Fourier's law for conduction of energy $q_x = -k\partial T/\partial x$)

$$\overline{vk'} = -\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial y}.$$

The diffusion by pressure is usually negligible.

Now we can write the *modelled* k equation

$$\frac{\partial \bar{U}k}{\partial x} + \frac{\partial \bar{V}k}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + \nu_t\left(\frac{\partial \bar{U}}{\partial y}\right)^2 - \varepsilon. \quad (71)$$

An exact equation for ε *can* be derived, but it is very complicated. From the exact equation we do get some information: as the k equation, the production term in the ε equations includes both mean flow gradients (such as $\partial\bar{U}/\partial y$) and turbulent quantities, and the destruction term includes only turbulent quantities. Thus we derive the modelled ε equation by looking at the modelled k equation. We write the right-hand side as

$$\frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right] + \frac{1}{\tau}\left(c_{\varepsilon 1}P_k - c_{\varepsilon 2}\varepsilon\right).$$

In the diffusion term we have simply added a constant which represent a turbulent Prandtl number for ε. Similarly two additional constants have been added for the source terms, one in front of the production term and one in front of the dissipation (destruction) term. Furthermore, a time scale has been added to make sure that correct dimensions are recovered. This time scale should be a turbulent one, and thus $\tau = k/\varepsilon$ [s]. We can now write the modelled ε equation

as

$$\frac{\partial \bar{U}\varepsilon}{\partial x}+\frac{\partial \bar{V}\varepsilon}{\partial y}=\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right]+\frac{\varepsilon}{k}\left[c_{\varepsilon 1}\nu_t\left(\frac{\partial \bar{U}}{\partial y}\right)^2-c_{\varepsilon 2}\varepsilon\right]. \tag{72}$$

Equations 70, 71 and 72 form the $k-\varepsilon$ model (see 3.5.2 in V & M and Section 3.3 in LD). It includes five universal constants: c_μ, $c_{\varepsilon 1}$, $c_{\varepsilon 2}$, σ_k and σ_ε.

- Boundary layer flow:

  k equation $\Rightarrow$ c_μ

  ε equation $\Rightarrow$ $c_{\varepsilon 1}=\text{func}\{c_{\varepsilon 2},\sigma_\varepsilon,c_\mu\}$

- Decaying turbulence

$$\left.\begin{array}{ll}\bar{U}dk/dx & =-\varepsilon \\ \bar{U}d\varepsilon/dx & =-c_{\varepsilon 2}\varepsilon^2/k \\ \exp: & k\propto x^{-m}\end{array}\right\}\Rightarrow c_{\varepsilon 2}=\text{func}(m)$$

- σ_k and σ_ε determined from computer optimization

The c_μ constant

In the log-law of the boundary layer we know from experiments that the production term P_k and the dissipation ε in the k equation are much larger than the other terms (see Section 3.1 in LD), and consequently the k equation reads

$$0=P_k-\varepsilon.$$

The turbulence is in *local equilibrium*, since transport effects are negligible. The production is equal to $P_k = \nu_t(\partial\bar{U}/\partial y)^2$. Multiply the equation above by ν_t so that

$$\left(\nu_t \frac{\partial \bar{U}}{\partial y}\right)^2 = \nu_t \varepsilon.$$

The Boussinesq assumption

$$-\overline{uv} = \nu_t \frac{\partial \bar{U}}{\partial y}$$

gives

$$\left(\overline{uv}\right)^2 = \nu_t \varepsilon.$$

Insert the expression for the turbulent viscosity (Eq. 70) so that

$$\left(\overline{uv}\right)^2 = c_\mu k^2 \;\Rightarrow c_\mu^{1/2} = \frac{|\overline{uv}|}{k}.$$

From experiments it is found that in the log-region, the ratio between $|\overline{uv}|$ and k is 0.3, which gives $c_\mu = 0.09$.

The $c_{\varepsilon 2}$ constant

This constant is obtained from grid-decaying turbulence. Turbulence is generated when free flow (U_∞) goes through a grid which generates mean-flow gradients $\partial U/\partial x, \partial U/\partial z$, which in turn, via the production term, generates turbulence. Sufficiently far downstream the velocity flow is constant (i.e. the velocity gradients are zero), so that the production term in the k and ε equations are negligible. The diffusion terms are also small, and hence we can write the

(modelled) k and ε equations as

$$U_\infty \frac{dk}{dx} = -\varepsilon$$
$$U_\infty \frac{d\varepsilon}{dx} = -c_{\varepsilon 2}\frac{\varepsilon^2}{k}.$$

Let's assume that k decays as $k = Cx^{-m}$. From the k equation we obtain $\varepsilon = -CU_\infty m x^{-m-1}$. Insert this into the ε equation

$$U_\infty C U_\infty m(m+1)x^{-m-2} = -c_{\varepsilon 2}\frac{\left(CU_\infty m x^{-m-1}\right)^2}{Cx^{-m}}$$
$$\Rightarrow m(m+1)x^{-m-2} = -c_{\varepsilon 2}\frac{m^2 x^{-2m-2}}{x^{-m}},$$

which gives $c_{\varepsilon 2} = (m+1)/m$. Experiments give $m = 1.25 \pm 0.06$, and $c_{\varepsilon 2} = 1.92$ is chosen.

The $c_{\varepsilon 1}$ constant

This constant is obtained by looking at the ε equation in the log-region of a boundary layer, which reads

$$0 = c_{\varepsilon 1}\frac{\varepsilon}{k}P_k + \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) - c_{\varepsilon 2}\frac{\varepsilon^2}{k}. \tag{73}$$

Note that in the ε equation, the turbulent diffusion term is not negligible, whereas in the k it is. The reason is that while $\partial k/\partial y \simeq 0$ in the log region, the derivative of the length scale is not negligible. We start by introducing the usual approximations valid in the log region:

$$P_k = \varepsilon \simeq \frac{\mathcal{U}^3}{\ell} = \frac{u_*^3}{\kappa y}$$
$$\nu_t \simeq \mathcal{U}\ell = \kappa u_* y,\ k = c_\mu^{-1/2}u_*^2 \tag{74}$$

Let's start with the diffusion term (recall that κ, σ_ε and u_* are independent of y)

$$\frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) = \frac{\partial}{\partial y}\left[\frac{\kappa u_* y}{\sigma_\varepsilon}\frac{\partial}{\partial y}\left(\frac{u_*^3}{\kappa y}\right)\right] = \frac{u_*^4}{\sigma_\varepsilon}\frac{\partial}{\partial y}\left[y\frac{\partial}{\partial y}\left(\frac{1}{y}\right)\right] = \frac{u_*^4}{\sigma_\varepsilon}\frac{\partial}{\partial y}\left[y(-y^{-2})\right] = \frac{u_*^4}{\sigma_\varepsilon y^2}. \quad (75)$$

Insertion of Eqs. 74 and 75 into Eq. 73 gives

$$0 = (c_{\varepsilon 1} - c_{\varepsilon 2})\frac{u_*^3}{\kappa y}\frac{1}{c_\mu^{-1/2} u_*^2}\frac{u_*^3}{\kappa y} + \frac{u_*^4}{\sigma_\varepsilon y^2}$$

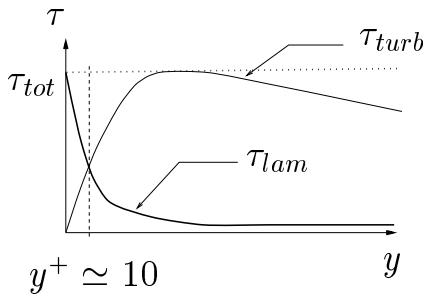
$$\Rightarrow c_{\varepsilon 1} = c_{\varepsilon 2} - \frac{\kappa^2}{c_\mu^{1/2}\sigma_\varepsilon}$$

The standard values of the coefficients are: $c_\mu = 0.09$, $c_{\varepsilon 1} = 1.44$, $c_{\varepsilon 1} = 1.44$, $\sigma_k = 1$, $\sigma_\varepsilon = 1.3$

Wall Functions

In industrial CFD wall functions are often used. In this way the CPU time can be reduced substantially. Using wall functions means that the boundary near a wall is not resolved, but the first node is located in the log-law region where $30 \leq y^+ \leq 100$ (the upper limit is dependent of the Reynolds number). The flow between the first node and the wall is supposed to be as in flat-plate boundary layer flow. This assumption is often well satisfied, but it many flow situations it is not true at all.

As mentioned above, the ratio between $|\overline{uv}|$ and k is 0.3. And, furtermore, the total (viscous plus turbulent) stress is constant from the wall out into the log-law region (see the figure below)



A wall-friction velocity u_* is defined from the wall-shear stress τ_w as

$$\tau_w = \rho u_*^2,$$

so that $u_*^2/k = 0.3 \equiv \sqrt{c_\mu}$. This gives us a condition for k at the first interior node (*not* the boundary node). The friction velocity u_* is obtained from the log-law (see Eq. 3.21 in V & M and Section 3.2 in LD)

$$\frac{\bar{U}_P}{u_*} = \frac{1}{\kappa} \ln(Ey^+)$$

where E is a constant, $y^+ = u_* y/\nu$, and $\bar{U}_P$ denotes the velocity component parallel to the wall, computed with our finite volume method. The equation is solved by rewriting it so that

$$u_* = \frac{\kappa \bar{U}_P}{\ln\left(Eu_* y/\nu\right)}.$$

A few iterations are usually sufficient to obtain u_*.

As mentioned above, in the log-region we can write the k equation as

$$0 = P_k - \varepsilon.$$

In the log-region we have from experiments that $-\overline{uv} \simeq u_*^2$ (see Fig. 2.1 in LD). The production term can thus be

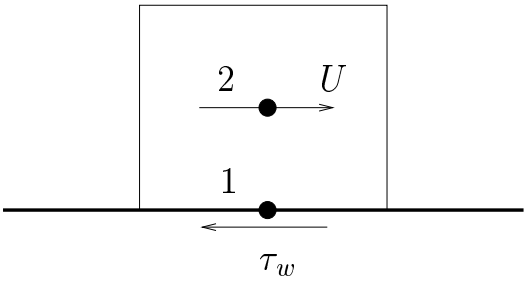
written

$$P_k = -\overline{uv}\frac{\partial \bar{U}}{\partial y} = u_*^2\frac{\partial \bar{U}}{\partial y} = \frac{u_*^3}{\kappa y}$$

where the velocity gradient was taken from the log-law above. In this way the dissipation is set at the first interior node as

$$\varepsilon = \frac{u_*^3}{\kappa y}$$

The velocity component normal to the wall is set to zero. For the component parallel to the wall we prescribe a force (cf. when in the temperature equation the heat flux is prescribed rather than the temperature itself). The force per unit area is the wall shear stress τ_w, which should have opposite sign to the velocity component, see the figure below (an exception is when the wall is moving, driving the flow).



The wall shear stress is put in the S_P part of the source term (remember that $S = S_P\Phi_P + S_U,\ S_P < 0$)

$$S_P = -\frac{\tau_w A}{\bar{U}_P} = -\frac{\kappa\rho u_* A}{\ln\left(Ey^+\right)}$$

where A denotes the wall area of the cell; the log-law was used to obtain the last expression in the above equation.

We can summarize the wall functions:

1. k is fixed at the near-wall node as $k = c_\mu^{-0.5} u_*^2$; the friction velocity u_* is obtained from the log-law
2. ε is fixed at the near-wall node as $\varepsilon = u_*^3/(\kappa y)$
3. The wall shear stress is used as a force-boundary condition for the velocity component parallel to the wall.

The $k-\omega$ Model

The $k-\omega$ model is a another two-equation model which is becoming more and more common. This model was originally developed by Wilcox [8]. In this model the k equation is used together with an ω equation. The variable ω is defined as

$$\omega = \frac{\varepsilon}{\beta^* k},$$

and its exact transport equation is derived from the exact k and ε equations (see Peng [5]). As can be seen from the definition of ω its physical meaning is an inverse time scale ([s^{-1}]). The $k-\omega$ model reads in boundary layer form (see Section 3.4 in LD)

$$\frac{\partial \rho \bar{U} k}{\partial x} + \frac{\partial \rho \bar{V} k}{\partial y} = \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_t}{\sigma_k^\omega}\right)\frac{\partial k}{\partial y}\right] + P_k - \beta^* \omega k$$

$$\frac{\partial \rho \bar{U} \omega}{\partial x} + \frac{\partial \rho \bar{V} \omega}{\partial y} = \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial y}\right] + \frac{\omega}{k}\left(c_{\omega 1} P_k - c_{\omega 2} \rho k \omega\right)$$

The turbulent viscosity is obtained from

$$\nu_t = \frac{k}{\omega}.$$