Turbulence Models

The Navier-Stokes equations can be solved numerically by resolving *all* turbulent scales. This, however, requires extremely large computer resources, and can only by carried out for low Re numbers. Sofar, it has been done for backward-facing step at $Re = 5000$ [3].

In the forseeable future, we have to rely to the Reynolds decomposition, where we employ turbulence models.

The Navier-Stokes equations for 1D, incompressible reads

$$\frac{d}{dx}(UU) = -\frac{1}{\rho}\frac{dP}{dx} + \frac{d}{dx}\left(\nu\frac{dU}{dx}\right), \tag{62}$$

where U is the instantaneous velocity.

In turbulent flow we divide the instantaneous velocity U into a mean (time-averaged) part $\bar{U}$ and a fluctuating part u so that (see Chapter 2.1 in LD, Chapter 3.3 in V & M, Fig. 3.1 in V & M)

$$U = \bar{U} + u. \tag{63}$$

This is called Reynolds decomposition. The time-averaged velocity is obtained from

$$\bar{U} = \frac{1}{2\Delta T}\int_{-T}^{T} U(\tau)d\tau.$$

Insert Eq. 63 into Eq. 62, and since $\overline{u} = 0$ we have the following relations: (see p. 50 in V & M)

$$\overline{\bar{U}u} = \overline{\bar{U}}\,\overline{u} = 0, \ \overline{\bar{U} + u} = \overline{\bar{U}} + \overline{u} = \bar{U}. \tag{64}$$

Time averaging gives

$$\frac{d}{dx}\overline{\left[(\bar{U}+u)(\bar{U}+u)\right]} = -\frac{1}{\rho}\frac{d}{dx}\overline{\left(\bar{P}+p\right)} + \frac{d}{dx}\left[\nu\frac{d}{dx}\overline{\left(\bar{U}+u\right)}\right]. \tag{65}$$

Using Eq. 64, the two terms on the right-hand side of Eq. 65 can immediately be written as

$$-\frac{1}{\rho}\frac{d}{dx}\overline{\left(\bar{P}+p\right)} = -\frac{1}{\rho}\frac{d\bar{P}}{dx}$$

$$\frac{d}{dx}\left[\nu\frac{d}{dx}\overline{\left(\bar{U}+u\right)}\right] = \frac{d}{dx}\left(\nu\frac{d\bar{U}}{dx}\right).$$

The left-hand side of Eq. 65 can, using Eq. 64, be written as

$$\frac{d}{dx}\overline{\left[(\bar{U}+u)(\bar{U}+u)\right]} = \frac{d}{dx}\overline{\left(\bar{U}^2+2\bar{U}u+u^2\right)} =$$

$$\frac{d}{dx}\overline{\left(\bar{U}^2+u^2\right)} = \frac{d}{dx}\left(\bar{U}^2+\overline{u^2}\right).$$

When this is inserted into Navier-Stokes equation we obtain the *Reynolds equations*. Below the steady, Reynolds equation for $\bar{U}$ in boundary layer flow is given

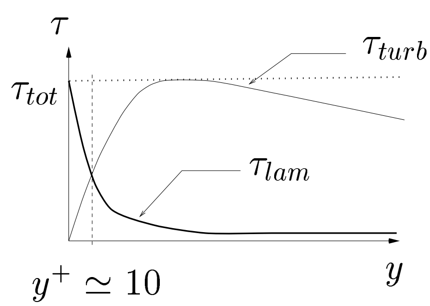
$$\frac{\partial}{\partial x}\left(\rho\bar{U}\bar{U}\right) + \frac{\partial}{\partial y}\left(\rho\bar{V}\bar{U}\right) = -\frac{\partial\bar{P}}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial\bar{U}}{\partial y} - \rho\overline{uv}\right). \tag{66}$$

In Table 3.1 in V & M the full three-dimensional equations are given on component form and in Section 2.1 in LD on tensor notation. On the right-hand side a new unknown term appear ($-\rho\overline{uv}$), which can be regarded as an additional stress due to the decomposition; it is called a *Reynolds stress*. Since it is unknown it must be *modeled*: we must introduce a turbulence model.

The total stress is now the sum of the viscous and the turbulent stress, i.e.

$$\tau_{tot} = \mu \frac{\partial \bar{U}}{\partial y} - \rho \overline{uv}.$$

Near the wall the turbulent stress goes to zero. In the fully turbulent region, the turbulent stress is several magnitudes larger than the viscous one. The total stress is constant from the wall out in the logarithmic region, see the figure below



Eddy-Viscosity Model

We'd like to model the turbulence by adding an additional viscosity (a turbulent viscosity μ_t) to the viscous one. We rewrite the right-hand side of Eq. 66 as

$$\frac{\partial}{\partial y}\left[(\mu + \mu_t)\frac{\partial \bar{U}}{\partial y}\right]. \tag{67}$$

Identification of Eqs. 66 and 67 gives

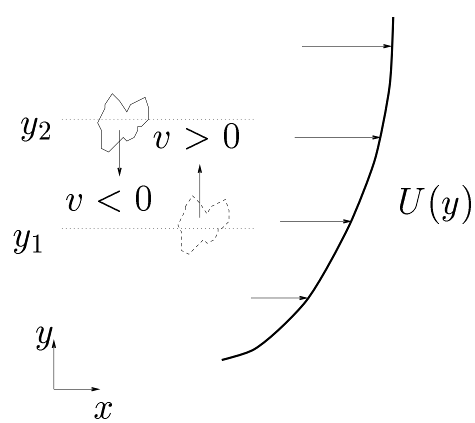
$$-\rho \overline{uv} = \mu_t \frac{\partial \bar{U}}{\partial y}. \tag{68}$$

This is called the *Boussinesq* assumption (cf. Section 3.5 in V & M and Section 2.2 in LD). Note that this is an *assumption*. The turbulent viscosity μ_t, unlike the laminar one μ, depends on the flow, and it can be written as a product of a turbulent length and velocity scale, i.e.

$$\mu_t \propto \rho \mathcal{U} \ell \tag{69}$$

Physical interpretation of $\overline{uv}$

Let's study the flow in a boundary layer where $\partial\bar{U}/\partial y > 0$. A fluid particle at level y_2 is moving downwards ($v < 0$) to level y_1, shown with solid line in the figure below. It is moving to a region where the velocity in average is smaller than from where it comes from, i.e. $\bar{U}(y_1) < \bar{U}(y_2)$. Thus, when it comes to level y_1 the fluid particle carries higher U momentum than its new surrounding, and hence $u >$. Consequently $\overline{uv} < 0$.

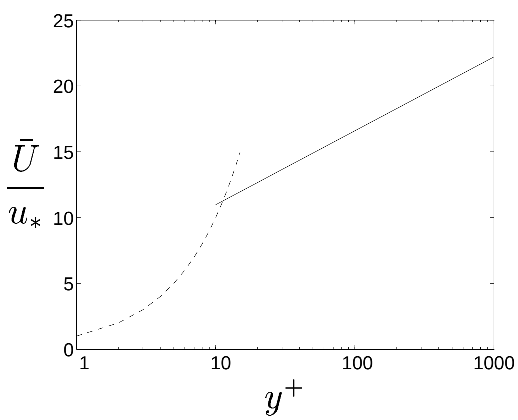


If we now take a particle moving from level y_1 to level y_2, shown with dashed line in the figure above. It's moving upwards ($v > 0$). When is comes to the new level, it carries lower U momentum than its new surrounding, i.e. $u = U(y_1) - U(y_2) < 0$. Thus, $\overline{uv} < 0$.

If we would change the sign of the gradient of $\partial\bar{U}/\partial y$, we'd find the opposite, i.e. $\overline{uv} > 0$. We have found that the velocity gradient $\partial\bar{U}/\partial y$ and the turbulent shear stress $\overline{uv}$ have opposite signs. The reason for this can be explained by the production term in the k equation (see a later lecture). This is mostly true, but not always. One exception is the wall jet in the region of the velocity maximum, where the velocity gradient and the turbulent shear stress have the same sign.

The log-law

Below the velocity profile in a boundary layer is shown (cf. 3.11 in V & M)



We have three main regions:

1. The viscous sublayer: $y^+ \lesssim 5$ where $u^+ = y^+$ ($u^+ = \bar{U}/u_*$, $y^+ = u_* y/\nu$);
2. The logarithmic region: $30 \lesssim y^+ \lesssim 100$ (the upper limit depend on Re number) where $u^+ = \ln(Ey^+)/\kappa$ ($E = 9$, $\kappa = 0.41$);
3. Wake region: above (higher y^+) the logarithmic region

The buffert region $5 \lesssim y^+ \lesssim 30$ is an intermediate region between the viscous sublayer and the logarithmic region, where neither the linear law nor the log law are valid.

The k equation

An exact equation can be derived for the turbulent, kinetic energy k which is defined as

$$k = \frac{1}{2}\overline{u_i u_i}, \quad \text{on component form} : k = \frac{1}{2}\left(\overline{u^2} + \overline{v^2} + \overline{w^2}\right).$$

It is derived from Navier-Stokes as follows (for greater detail, see Section 2.4 in LD):

1. N-S: $\partial \rho U_i / \partial t + (\rho U_j U_i)_{,j} \ldots$
2. Time averaged N-S: $\partial \rho \bar{U}_i / \partial t + (\rho \bar{U}_j \bar{U}_i)_{,j} \ldots$
3. Subtract Eq. 1 from Eq.2, multiply by u_i and time average.

The exact k equation reads on boundary-layer form

$$\underbrace{\frac{\partial \rho k}{\partial t} + \frac{\partial \rho \bar{U} k}{\partial x} + \frac{\partial \rho \bar{V} k}{\partial y}}_{I} =$$

$$\underbrace{-\rho\overline{uv}\frac{\partial \bar{U}}{\partial y}}_{II} \underbrace{-\frac{\partial}{\partial y}\left[\overbrace{\overline{pv} + \frac{1}{2}\rho\overline{v u_i u_i}}^{a} - \overbrace{\mu\frac{\partial k}{\partial y}}^{b}\right]}_{III} - \underbrace{\mu\overline{u_{i,j} u_{i,j}}}_{IV}.$$

The terms have the following physical meaning:

I Unsteady and convective term;

II Production term; as discussed on p. 80 this term is mostly positive, since the velocity gradient and the shear stress mostly have different signs. This term is responsible of extracting energy from the mean flow, and this term is large for the large, energy-containing eddies (Region I in the figure on p. 73).

IIIa Turbulent diffusion by velocity fluctuations and pressure; dominated by large scales.

IIIb Viscous diffusion.

IV Dissipation. This term is large for small scales (Region III in the figure on p. 73).

In the log-region the two largest terms are the production and the dissipation, and they are in balance, i.e.

$$-\overline{uv}\frac{\partial \bar{U}}{\partial y} = \nu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}.$$

If we estimate $\overline{uv}$ employing the Boussinesq assumption (Eq. 68) and the estimate of the turbulent viscosity in Eq. 69 we get

$$\frac{\mathcal{U}\ell}{\nu}\left(\frac{\partial \bar{U}}{\partial y}\right)^2 = \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}.$$

For fully turbulent flow, the turbulent Reynolds number $Re_\ell = \mathcal{U}\ell/\nu \gg 0$. Hence

$$\overline{\left|\frac{\partial u_i}{\partial x_j}\right|} \gg \left|\frac{\partial \bar{U}}{\partial y}\right|.$$

In the previous Chapter we emphasised that the dissipation takes place at the smallest scales, the main reason being that the dissipation term is large for these scales, since the velocity gradient is large for these scales, as we've just indicated.