FREQUENCY DOMAIN ANALYSIS OF DYNAMIC SYSTEMS

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Contents

Norms

- Consider a linear continuous time invariant system with transfer function

$$H(s) = C(sI - A)^{-1}B + D \in \mathbb{C}^{r \times m}$$

where $j\omega \in \mathcal{D}(H)$ for all $\omega \in \mathbb{R}$. The impulse response is

$$h(t) = Ce^{At}B + D\delta(t) \in \mathbb{R}^{r \times m}$$

$$\Downarrow$$

Norms based on either time response $h(t)$ or frequency response $H(j\omega)$

H_2 norm

- Using the Parseval's theorem it is verified that

Fact

The following equality holds

$$\int_0^\infty \mathrm{Tr}(h(t)'h(t))\ dt = \frac{1}{\pi}\int_0^\infty \mathrm{Tr}(H(j\omega)^\sim H(j\omega))d\omega$$

- It is important to notice that :
  - the above equality still holds if $h(t)$ and $H(s)$ are replaced by $h(t)'$ and $H(s)'$ respectively. That is

$$(A, B, C, D) \Longrightarrow (A', C', B', D')$$

H_2 norm

- Let $\mathcal{S}$ be a linear time invariant system with transfer function (or state space representation) defined by matrices of compatible dimensions, denoted as $\mathcal{S} = (A, B, C, D)$.

Lemma (H_2 norm)

The H_2 norm of system $\mathcal{S}$ is given by

$$\|\mathcal{S}\|_2^2 := \int_0^{\infty} \mathrm{Tr}(h(t)'h(t))\ dt = \frac{1}{\pi}\int_0^{\infty} \mathrm{Tr}(H(j\omega)^{\sim} H(j\omega))d\omega$$

- The above equalities hold whenever $\mathcal{S}$ is asymptotically stable, that is

$$j\omega \in \mathcal{D}(H)\ ,\ \forall\ \omega \in \mathbb{R}$$

H_2 norm

- Important : $\|\mathcal{S}\|_2 < \infty$ if and only if $D = 0$. The H_2 norm is finite only for strictly proper systems. From the previous Lemma we have

$$\|\mathcal{S}\|_2^2 = \int_0^\infty \mathrm{Tr}\left(B' e^{A' t} C' C e^{At} B\right) dt + \mathrm{Tr}(B' C' D) + \mathrm{Tr}(D' C B) + \mathrm{Tr}(D' D) \int_0^\infty \delta(t)^2 dt$$

  - From Parseval's theorem :

$$\int_0^\infty \delta(t)^2 dt = \frac{1}{\pi} \int_0^\infty d\omega = +\infty$$

  Then, $\|\mathcal{S}\|_2$ finite requires $D = 0$.

H_2 norm

- For strictly proper time invariant systems the H_2 norm is calculated as follows :

Lemma (H_2 norm calculation)

The following hold :

- $\|\mathcal{S}\|_2^2 = \mathrm{Tr}(B'P_oB)$ *where P_o is the observability gramian :*

$$P_o = \int_0^\infty e^{A't}C'Ce^{At}dt$$

- $\|\mathcal{S}\|_2^2 = \mathrm{Tr}(CP_cC')$ *where P_c is the controllability gramian :*

$$P_c = \int_0^\infty e^{At}BB'e^{A't}dt$$

H_2 norm

- Other possibilities for H_2 norm calculation using LMI :
  - Bounding the observability gramian

$$\|\mathcal{S}\|_2^2 = \inf_{X>0} \{\mathrm{Tr}(B'XB) \ : \ A'X + XA + C'C < 0\}$$
$$= \sup_{Y>0} \{\mathrm{Tr}(B'YB) \ : \ A'Y + YA + C'C > 0\}$$

  - Bounding the controllability gramian

$$\|\mathcal{S}\|_2^2 = \inf_{X>0} \{\mathrm{Tr}(CXC') \ : \ AX + XA' + BB' < 0\}$$
$$= \sup_{Y>0} \{\mathrm{Tr}(CYC') \ : \ AY + YA' + BB' > 0\}$$

Using LMI solvers, all these problems provide the H_2 norm of the system $\mathcal{S}$ within a precision defined by the designer.

H_∞ norm

- Let $V \in \mathbb{C}^{r\times m}$ be a complex matrix. The matrix

$$Q = V^{\sim} V \in \mathbb{C}^{m\times m}$$

is Hermitian and positive semidefinite, that is :

  - $Q^{\sim} = Q$ and $v^{\sim} Q v \geq 0$ for all $v \in \mathbb{C}^m$.
  - The eigenvalues of Q satisfies $\lambda_i(Q) \geq 0$ for all $i = 1, \cdots, m$.
  - The quantities

$$\sigma_i(V) := \sqrt{\lambda_i(Q)} = \sqrt{\lambda_i(V^{\sim} V)}\ , \quad i = 1, \cdots, m$$

    are the singular values of V.
  - The quantity

$$\|V\|_\infty := max_{i=1,\cdots,m} \sigma_i(V) := \sigma_M(V)$$

    is the ∞-norm of V. Moreover $\|V\|_\infty = \|V'\|_\infty$.

H_∞ norm

- Let $\mathcal{S}$ be a linear time invariant system with transfer function (or state space representation) defined by matrices of compatible dimensions, denoted as $\mathcal{S} = (A, B, C, D)$.

Lemma (H_∞ norm)

The H_∞ norm of system $\mathcal{S}$ is given by

$$\|\mathcal{S}\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_M(H(j\omega))$$

- The above equality holds whenever $\mathcal{S}$ is asymptotically stable, that is

$$j\omega \in \mathcal{D}(H) \ , \ \forall \ \omega \in \mathbb{R}$$

H_∞ norm

- Time domain interpretation : Consider the state space representation of $H(s)$ as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bw(t)\ ,\ x(0) = 0\\ z(t) &= Cx(t) + Dw(t)\end{aligned}$$

where $0 \in \mathcal{D}(\hat{w})$. The output $\hat{z}(s) = H(s)\hat{w}(s)$ gives

$$\begin{aligned}\int_0^\infty z(t)'z(t)dt &= \frac{1}{\pi}\int_0^\infty \underbrace{\hat{w}(j\omega)^\sim H(j\omega)^\sim H(j\omega)\hat{w}(j\omega)}_{\hat{z}(j\omega)^\sim \hat{z}(j\omega)} d\omega\\ &\leq \|\mathcal{S}\|_\infty^2 \int_0^\infty w(t)'w(t)dt\end{aligned}$$

Fact : $\|\mathcal{S}\|_\infty \leq \gamma \iff \|z(t)\|_2 \leq \gamma \|w(t)\|_2$

H_∞ norm

- Using the quadratic Lyapunov function $v(x) = x'Px$ with $P > 0$ and imposing

$$\dot{v}(x(t)) \leq -z(t)'z(t) + \gamma^2 w(t)'w(t) \ , \quad \forall t \geq 0$$

for some $\gamma \geq 0$, the time integration from 0 to $+\infty$ provides

$$\int_0^\infty z(t)'z(t)dt - \gamma^2 \int_0^\infty w(t)'w(t)dt \leq 0$$

yielding the conclusion that $\|\mathcal{S}\|_\infty \leq \gamma$. On the other hand, taking into account the state space representation of $\mathcal{S}$ we get

$$\dot{v}(x(t)) = (Ax(t) + Bw(t))'Px(t) + x(t)'P(Ax(t) + Bw(t))$$

H_∞ norm

- For a linear time invariant system with transfer function (or state space representation) defined by matrices of compatible dimensions, denoted as $\mathcal{S} = (A, B, C, D)$, the H_∞ norm is calculated as follows :

Lemma (H_∞ norm calculation)

The H_∞ norm of system $\mathcal{S}$ is given by

$$\|\mathcal{S}\|_\infty^2 := \inf_{\mu, X>0} \left\{ \mu \ : \ \begin{bmatrix} A'X + XA & XB & C' \\ \bullet & -\mu I & D' \\ \bullet & \bullet & -I \end{bmatrix} < 0 \right\}$$

This problem can be solved with no big difficulty since it is expressed by an LMI with respect to the variables $\mu \in \mathbb{R}$ and $X = X' \in \mathbb{R}^{n \times n}$.

KYP Lemma

- Is one of the most general results on frequency domain. Consider a transfer function $H(s)$ with state space representation $\mathcal{S} = \{A, B, C, D\}$, $\det(j\omega I - A) \neq 0, \forall\ \omega \in \mathbb{R}$ and a symmetric matrix Π of compatible dimension.

Lemma (KYP Lemma)

The transfer function $H(s)$ satisfies the constraint

$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^{\sim} \Pi \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R}$$

if and only if there exists $P = P'$ such that

$$\begin{bmatrix} A'P + PA & PB \\ B'P & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}' \Pi \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} < 0$$

KYP Lemma

- Sufficiency is simple to prove from the matrix function

$$\mathcal{A}(s) := (sI - A)^{-1}$$

which allows us to verify that :

  - $A\mathcal{A}(s) = -I + s\mathcal{A}(s)$ for all $s \in \mathbb{C}$.
  - For any $P = P' \in \mathbb{R}^{n\times n}$ the matrix function

$$\mathcal{Q}(\omega) := \mathcal{A}(j\omega)^{\sim}(A'P + PA)\mathcal{A}(j\omega) + \mathcal{A}(j\omega)^{\sim}P + P\mathcal{A}(j\omega)$$

    satisfies $\mathcal{Q}(\omega) = 0$ for all $\omega \in \mathbb{R}$.
  - The following factorization of $H(s)$ holds

$$\begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} \mathcal{A}(s)B \\ I \end{bmatrix} = \begin{bmatrix} I \\ H(s) \end{bmatrix}, \quad \forall s \in \mathbb{C}$$

KYP Lemma

- Multiplying the second inequality of the KYP Lemma to the left by $[B'\mathcal{A}(j\omega)^\sim \ I]$ and to the right by its transpose, from the previous results we obtain

$$B'\mathcal{Q}(\omega)B + \begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^\sim \Pi \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R}$$

  and the first inequality of the KYP Lemma holds due to the fact that $\mathcal{Q}(\omega) = 0$ for all $\omega \in \mathbb{R}$.
- The necessity states that if the first inequality of the KYP Lemma holds for some matrix Π then the second one also holds for the same Π for some matrix $P = P'$.
- If A is asymptotically stable then $P = P' > 0$ can be included with no loss of generality whenever $C'\Pi_{22}C \geq 0$.

Comparison

- It is seen that

$$\|H(s)\|_\infty < \gamma \iff \Pi = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}$$

The celebrated Small Gain Theorem is a mere particular case of the KYP Lemma which provides

$$\|H(s)\|_\infty < \gamma \iff \begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix} < 0$$

for some $P > 0$. The Schur Complement gives the previous LMI for H_∞ calculation with $\mu = \gamma^2$.

Locus in the s-space

- The is no difficulty to impose several constraints on $H(j\omega)$ characterized by different matrices $\Pi_i, i = 1, \cdots, N$. From the KYP Lemma the inequalities

$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^{\sim} \Pi_i \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R}$$

holds for all $i = 1, \cdots, N$ if and only if there exist matrices $P_i = P_i'$ such that

$$\begin{bmatrix} A'P_i + P_iA & P_iB \\ B'P_i & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}' \Pi_i \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} < 0$$

holds for all $i = 1, \cdots, N$.

Locus in the s-space

- Examples for SISO systems : $H(j\omega) \in \mathbb{C}$ for each $\omega \in \mathbb{R}$
  - Circle : $|H(j\omega)| < \gamma$ for all $\omega \in \mathbb{R}$

$$H(j\omega)^* H(j\omega) < \gamma^2 \iff \Pi = \begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix}$$

  - Right hand side of C : $\mathrm{Re}(H(j\omega)) > 0$ for all $\omega \in \mathbb{R}$

$$\Pi = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

  - Strip : $\alpha < \mathrm{Re}(H(j\omega)) < \beta$ for all $\omega \in \mathbb{R}$

$$\Pi_1 = \begin{bmatrix} -2\beta & 1 \\ 1 & 0 \end{bmatrix}, \ \Pi_2 = \begin{bmatrix} 2\alpha & -1 \\ -1 & 0 \end{bmatrix}$$

State feedback

- Consider the linear time invariant system :

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= Cx(t) + D_1 w(t) + D_2 u(t) \\ u(t) &= Kx(t) \end{aligned}$$

The closed loop transfer function is given by

$$H(s) = (C + D_2 K)(sI - (A + B_2 K))^{-1} B_1 + D_1$$

The determination of matrix K using LMIs is related to the transfer function $G(s) = H(s)'$, given by

$$G(s) = B_1'(sI - (A + B_2 K)')^{-1}(C + D_2 K)' + D_1'$$

State feedback

- Given matrices $\Pi_1, \cdots, \Pi_N$, the inequalities

$$\begin{bmatrix} I \\ G(j\omega) \end{bmatrix}^{\sim} \Pi_i \begin{bmatrix} I \\ G(j\omega) \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R}$$

for $i = 1, \cdots, N$ are satisfied if there exists a matrix $P = P'$ such that

$$\begin{bmatrix} (A + B_2 K)P + P(A + B_2 K)' & P(C + D_2 K)' \\ (C + D_2 K)P & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & I \\ B_1' & D_1' \end{bmatrix}' \Pi_i \begin{bmatrix} 0 & I \\ B_1' & D_1' \end{bmatrix} < 0$$

holds for all $i = 1, \cdots, N$.

State feedback

- Assuming that all matrices $\Pi_1, \cdots, \Pi_N$ are such that

$$\begin{bmatrix} 0 & B_1 \end{bmatrix} \Pi_i \begin{bmatrix} 0 \\ B_1' \end{bmatrix} \geq 0 \ , \ i = 1, \cdots, N$$

  the asymptotical stability of the closed loop system $A + B_2 K$ is assured by the linear constraint $P > 0$.

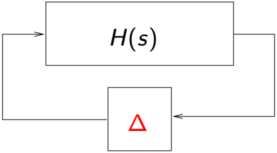
- The previous inequalities expressed in terms of the matrix variables $(P > 0, K)$ are converted into LMIs with respect to the matrix variables $(P > 0, L)$ where

$$K = LP^{-1}$$

  Since P can not depend on the index $i = 1, \cdots, N$ the necessity part of the KYP Lemma is lost. For $N = 1$ the necessity obviously holds.

Stability

- Systems with special structure - Linear part plus feedback



State space realization

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bw(t)\\ z(t) &= Cx(t) + Dw(t)\\ w(t) &= \Delta z(t)\ ,\ \Delta \in \Xi\end{aligned}$$

Ξ includes *a priori* information about the structure of Δ

Stability

- Assuming that

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}' \Pi \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \, , \quad \forall \, \Delta \in \Xi$$

then multiplying to the left by z' and to the right by z and using the fact that $w = \Delta z$ we conclude that

$$\begin{bmatrix} w \\ z \end{bmatrix}' \Pi \begin{bmatrix} w \\ z \end{bmatrix} \geq 0$$

for all (w, z) of appropriate dimensions such that $w = \Delta z$.

- On the other hand, considering the Lyapunov function

$$v(x) = x'Px \, , \quad P = P' > 0$$

Stability

and imposing that its time derivative along any trajectory of the system under consideration satisfies

$$\dot{v}(x) + \begin{bmatrix} w \\ z \end{bmatrix}' \Pi \begin{bmatrix} w \\ z \end{bmatrix} < 0$$

asymptotic stability is preserved for all $\Delta \in \Xi$. The key point on this algebraic manipulation is that the equality

$$\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

and the KYP Lemma provide the next important result.

Stability

Lemma (Stability)

Assume that $0 \in \Xi$. The previous feedback structure is stable for all $\Delta \in \Xi$ provided there exists a symmetric matrix Π such that

$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^{\sim} \Pi \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R}$$

and

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}' \Pi \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0\,, \quad \forall\, \Delta \in \Xi$$

- The assumption $0 \in \Xi$ is necessary since $P = P'$ must be positive definite in order to be used in the Lyapunov function.
- Both constraints are convex (linear) with respect to Π.

Stability

- Important aspects: Some situations can be dealt with no particular difficulty :
  - The classical H_∞ stability condition is obtained if Π is imposed to be
$$\Pi = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}$$
for some $\gamma > 0$. Asymptotic stability is preserved whenever
$$\|H(s)\|_\infty < \gamma \ , \quad \|\Delta\|_\infty \leq \gamma^{-1}$$
  - Matrix Π can be parameter dependent, that is $\Pi = \Pi(\Delta)$.
  - From the KYP Lemma, matrix P can also be parameter dependent, that is $P = P(\Delta)$.
  - For polytopic systems with $\Xi = \mathrm{co}\{\Delta_1, \cdots, \Delta_N\}$ matrices $(P_i > 0, \Pi_i)$ can be used.

Problems

1. Consider the following asymptotically stable transfer functions
   - $H(s) = \frac{(s+2)}{(s^2+2s+2)(s+1)}$
   - $H(s) = \frac{(s-2)}{(s^2+2s+2)(s+1)}$
   - $H(s) = \frac{(s-2)^2}{(s^2+2s+2)(s+1)}$

   Determine the H_2 norm of each transfer function using gramians and a numerical routine of LMIsolver.
2. Consider the following asymptotically stable transfer functions
   - $H(s) = \frac{(s+2)}{(s^2+2s+2)(s+1)}$
   - $H(s) = \frac{(s-2)}{(s^2+2s+2)(s+1)}$
   - $H(s) = 1 + \frac{(s-2)^2}{(s^2+2s+2)(s+1)}$

   Determine the H_∞ norm of each transfer function using the singular value diagram and a numerical routine of LMIsolver.

Problems

3. Consider a linear plant with transfer function $H(s)$ plus a nonlinear feedback $w(t) = \Delta(z(t))$. Show that asymptotical stability is preserved whenever

$$\|H(s)\|_\infty < \gamma \ , \quad \Delta(z)'\Delta(z) \le \gamma^{-2} z'z$$

4. Use the solution of the previous problem to verify that the origin for a SISO system defined by

$$H(s) = \frac{(s^2+3s+3)(s-1)}{(s+1)^3} \ , \quad \Delta(z) = \begin{cases} 1 - e^{-z/4}, & z \ge 0 \\ -1 + e^{z/4}, & z \le 0 \end{cases}$$

is globally asymptotically stable.

Problems

5. Consider a linear plant with transfer function $H(s)$ plus a linear feedback $w(t) = -\delta z(t)$ where $\delta \in \mathbb{R}$. For $H(s)$ given by

$$H(s) = \frac{(s-1)(s-2)}{(s+1)^2(s+4)}$$

determine the upper bound δ_{max} such that asymptotic stability is preserved for all $0 \leq \delta \leq \delta_{max}$ using :

- The H_∞ theory
- The Routh criterion.

Compare the obtained results from the root locus plot (with respect to δ) of the closed loop system.

Problems

6. Consider the function

$$\hat{f}(s) = \frac{(1 - e^{-sT})}{s}$$

- Is it possible to determine $\|\hat{f}(s)\|_\infty$? Why ?
- In the affirmative case, determine $\|\hat{f}(s)\|_\infty$.
- Is it possible to determine $\left\|\frac{e^{-Ts}}{s}\right\|_\infty$?

7. Consider a time delay system with characteristic equation $P(s) + \kappa e^{-Ts} = 0$ where $\kappa, T \geq 0$. Using the previous result show that asymptotic stability is preserved whenever

$$T \left\| \frac{\kappa s}{P(s) + \kappa} \right\|_\infty < 1$$

Problems

8. Consider $H(s)$ and $G(s)$ two asymptotically stable transfer functions. Show that the following relations hold :
   - $\|H(s)G(s)\|_2 \leq \|H(s)\|_\infty \|G(s)\|_2$
   - $\|H(s)G(s)\|_\infty \leq \|H(s)\|_\infty \|G(s)\|_\infty$
9. Verify the results of the previous problem for

$$H(s) = \frac{(s+1)^4}{(s+2)^5}\ , \quad G(s) = \frac{(s-1)}{(s+1)^2}$$

10. Determine numerically the values of $\|G(s)\|_2$ and $\|G(s)\|_\infty$ for the transfer function

$$G(s) = \frac{e^{-Ts}}{s^3 + 4s^2 + 4s + \kappa e^{-Ts}}$$

defined with $\kappa = 1/2$ and $T = 2$.