FREQUENCY DOMAIN ANALYSIS OF DYNAMIC SYSTEMS

JOSÉ C. GEROMEL

DSCE / School of Electrical and Computer Engineering
UNICAMP, CP 6101, 13083 - 970, Campinas, SP, Brazil,
geromel@dsce.fee.unicamp.br

Campinas, Brazil, August 2006

Contents

Lur'e systems

- Lur'e systems are those presenting the following state space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \\ w(t) &= -\phi(z(t)) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in R^m$, $z(t) \in \mathbb{R}^r$ and

$$\phi(\cdot) \ : \ \mathbb{R}^r \rightarrow \mathbb{R}^m$$

is a vector-valued nonlinear function belonging to some set. The above model is a linear system with impulse response $\mathcal{L}(h(t)) = C(sI - A)^{-1}B + D$ with a nonlinear feedback defined by $w(t) = -\phi(z(t))$. The linear part is supposed to be asymptotically stable.

Passivity

- Consider the linear part of a previously defined Lur'e system with $r = m$. Passivity and Positive Realness are equivalent concepts.

Definition (Time domain characterization)

It is said to be Passive if for $x(0) = 0$ the following inequality holds

$$\int_0^T z(t)' w(t) dt \geq 0 \ , \quad \forall \ T \geq 0$$

Definition (Frequency domain characterization)

Its transfer function is Positive Real. That is, it satisfies

$$H(j\omega)^{\sim} + H(j\omega) \geq 0 \ , \quad \forall \ \omega \in \mathbb{R}$$

Passivity

- Passivity can be verified with LMIs. Using the Lyapunov function $v(x) = x'Px$ with $P = P' > 0$, from the time domain characterization we have that

$$\int_0^T (w(t)'z(t) + z(t)'w(t))dt > v(x(T)) > 0$$

implies (strictly) passivity. Hence, with $x(0) = 0$ we have to impose for all $(x, w) \neq (0, 0)$ that

$$\dot{v}(x) < w'z + z'w$$

$$\Uparrow$$

$$\underbrace{(Ax + Bw)'}_{\dot{x}'} Px + x'P \underbrace{(Ax + Bw)}_{\dot{x}} < w' \underbrace{(Cx + Dw)}_{z} + \underbrace{(Cx + Dw)'}_{z'} w$$

Passivity

- Main result on Passivity and Positive Real transfer functions :

Theorem (Passivity and Positive Real)

The transfer function $H(s) = C(sI - A)^{-1}B + D$ is (strictly) Positive Real if and only if there exists $P = P' > 0$ such that

$$\begin{bmatrix} A'P + PA & PB - C' \\ B'P - C & -D - D' \end{bmatrix} < 0$$

- Sufficiency follows from the previous quadratic Lyapunov function.
- Necessity follows from the KYP Lemma.

Passivity

- Indeed, the transfer function $H(s)$ is positive real whenever

$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^{\sim} \Pi \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} < 0 \,, \quad \forall\, \omega \in \mathbb{R}$$

  where the multiplier Π is given by

$$\Pi = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$$

  Hence, the KYP Lemma provides the necessity part of the previous theorem.

- Passivity is more restrictive than simple asymptotical stability. However, stability is preserved for all $\phi(\cdot)$ such that $\phi(0) = 0$ and $w'z = -\phi(z)'z \leq 0$.

Passivity

- Important remarks :
  - The previous theorem implies $D + D' > 0$ that is, $H(s)$ must be proper. This is required for extended passivity. Passivity (not extended) is imposed by making $D \to 0$ and

    $$A'P + PA < 0\ ,\ \ PB = C'\ ,\ \ P > 0$$

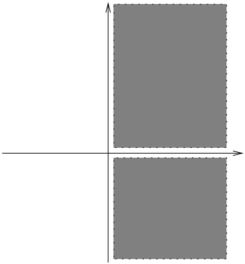
    in this case, the LMI given before is not strict.
  - For SISO systems, Passivity is equivalent to

    $$\mathrm{Re}(H(j\omega)) \geq 0\ ,\quad \forall \omega \in \mathbb{R}$$

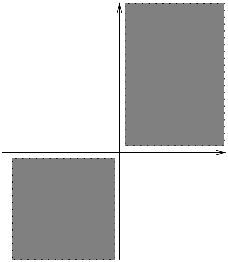
  - For SISO systems, the graphic of the nonlinear function $\phi(z)\ :\ \mathbb{R} \to \mathbb{R}$ with respect to $z \in \mathbb{R}$ must belong to the first and third quadrants.

Passivity

- For SISO systems stability is preserved whenever the locus of the linear part $H(s)$ and the graphic of the nonlinear part are as indicated :



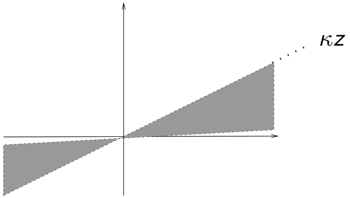
$\mathrm{Im}(H(j\omega)) \times \mathrm{Re}(H(j\omega))$



$\phi(z) \times z$

Popov criterion

- For simplicity we only consider SISO systems. The basic stability criterion applies to Lur'e systems with $r = m = 1$. The number of states n is arbitrary. The nonlinear function:
  - satisfies the condition $\phi(0) = 0$.
  - belongs to the sector $(0, \kappa > 0)$ that is $(\phi(z) - \kappa z)\phi(z) \leq 0$.



$\phi(z) \times z$

Popov criterion

- Quadratic Lyapunov function : Consider $v(x) = x'Px$ with $P = P' > 0$ and impose

$$\dot{v}(x) < (\phi(z) - \kappa z)'\phi(z) + \phi(z)'(\phi(z) - \kappa z)$$

Taking into account that $w = -\phi(z)$, this inequality is enforced by the existence of $P > 0$ such that

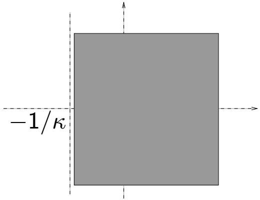
$$\begin{bmatrix} A'P + PA & PB - \kappa C' \\ B'P - \kappa C & -(I + \kappa D)' - (I + \kappa D) \end{bmatrix} < 0$$

From the Passivity Theorem this is possible if and only if the transfer function $G(s) := \kappa H(s) + I$ is strictly positive real.

Popov criterion

- Hence global asymptotic stability is preserved whenever

$$\mathrm{Re}(H(j\omega)) > -\frac{1}{\kappa}\ , \quad \forall\, \omega \in \mathbb{R}$$



$\mathrm{Im}(H(j\omega)) \times \mathrm{Re}(H(j\omega))$

- This condition is more stringent than the Nyquist criterion applied to the linear system defined by $\phi(z) = \beta z$ with $0 \leq \beta < \kappa$.

Popov criterion

- We now assume that the transfer function of the Lur'e system is strictly proper, that is $D = 0$. We introduce the Popov-Lyapunov function :

$$V(x) = x'Px + 2\theta \int_0^z \phi(\xi)d\xi$$

where $P = P' > 0$ and $\theta \geq 0$. It is important to keep in mind that, under our assumptions, this function is positive definite and radially unbounded. Hence is time derivative is given by

$$\begin{aligned} \dot{V}(x) &= \dot{x}'Px + x'P\dot{x} + \theta\phi(z)'\dot{z} + \theta\dot{z}'\phi(z) \\ &= (Ax + Bw)'(Px - \theta C'w) + \\ &\quad +(x'P - \theta w'C)(Ax + Bw) \end{aligned}$$

Popov criterion

- Finally, the constraint $\dot{V}(x) < (w + \kappa z)'w + w'(w + \kappa z)$ is assured from the existence of $P > 0$ and $\theta \geq 0$ satisfying the LMI

$$\begin{bmatrix} A'P + PA & PB - \theta A'C' - \kappa C' \\ B'P - \theta CA - \kappa C & -(I + \theta CB) - (I + \theta CB)' \end{bmatrix} < 0$$

Once again, applying the Passivity Theorem this is possible if and only if the transfer function

$$G(s) = (\kappa C + \theta CA)(sI - A)^{-1}B + (I + \theta CB)$$

is strictly positive real. The Popov criterion follows from the equality $A(sI - A)^{-1} = -I + s(sI - A)^{-1}$ which yields

$$G(s) = (\kappa + s\theta)H(s) + I$$

Popov criterion

- The previous calculations lead to the celebrated Popov criterion :

Theorem (Popov criterion)

Stability is preserved whenever there exits a scalar $\theta \geq 0$ such that

$$\mathrm{Re}\left(\left(1 + j\frac{\theta}{\kappa}\omega\right) H(j\omega)\right) > -\frac{1}{\kappa}$$

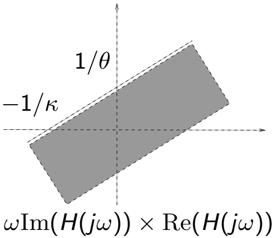
holds for all $\omega \in \mathbb{R}$.

- Notice that in the affirmative case the parameter $\theta \geq 0$ is readily determined and allows the construction of the Popov-Lyapunov function $V(x)$.
- The previous and more conservative stability condition is obtained by imposing $\theta = 0$ implying that $V(x) = v(x)$.

Frequency domain interpretation

- Simple algebraic manipulations show that the Popov stability condition can be rewritten as

$$\mathrm{Re}(H(j\omega)) - \frac{\theta}{\kappa}\omega \mathrm{Im}(H(j\omega)) > -\frac{1}{\kappa} \quad \forall \omega \in \mathbb{R}$$



- Notice that the complex plane used to apply the Popov criterion in not the same adopted by the Nyquist criterion.

Example

- Consider the asymptotically stable transfer function

$$H(s) = \frac{s+2}{s^4 + 6s^3 + 13s^2 + 14s + 6}$$

We have obtained the following values :

  - Nyquist criterion applied for $\phi(z)$ linear provides $\kappa \approx 17.36$.
  - Popov criterion with quadratic Lyapunov function applied for $\phi(z)$ nonlinear provides $\kappa \approx 9.38$.
  - Propov criterion applied for $\phi(z)$ nonlinear provides $\kappa \approx 17.33$.

- This example puts in evidence the quality of the result obtained from the Popov-Lyapunov function yielding the celebrated Popov criterion.

Sector optimization

- Using the LMI representation of the Popov criterion, there is no difficulty to solve the convex problem

$$\sup_{P>0,\theta>0,\kappa>0} \quad \kappa$$

subject to

$$\begin{bmatrix} A'P + PA & PB - \theta A'C' - \kappa C' \\ B'P - \theta CA - \kappa C & -(I + \theta CB) - (I + \theta CB)' \end{bmatrix} < 0$$

which provides :

- The parameters $P > 0$ and $\theta > 0$ of the function $V(x)$.
- The biggest sector defined by the optimal value of $\kappa > 0$ for which the Popov criterion holds.

State feedback

- Consider the Lur'e system with state feedback :

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= Cx(t) \\ u(t) &= Kx(t) \\ w(t) &= -\phi(z(t)) \end{aligned}$$

the goal is to determine the state feedback gain such that the sector defined by $\kappa > 0$ is maximized. The Popov criterion is used to check global asymptotic stability.
The key observation is that the Popov criterion can be rewritten in terms of $W := P^{-1} > 0$.

State feedback

- with products of variables θW and κW, that is

$$\begin{bmatrix} AW + WA' & B - \theta WA'C' - \kappa WC' \\ B' - \theta CAW - \kappa CW & -(I + \theta CB) - (I + \theta CB)' \end{bmatrix} < 0$$

However, from the definition of the new variable $K = LW^{-1}$ the closed-loop system is such that

$$\begin{aligned} AW &\rightarrow (A + B_2 K)W = AW + B_2 L \\ B &\rightarrow B_1 \end{aligned}$$

- The maximization of κ is an LMI with respect to matrices $W > 0$ and L for $\theta > 0$ and $\kappa > 0$ fixed.
- The optimization requires a two-dimensional parameter search.
- Setting $\theta = 0$, the nonlinear term κW remains.

Problems

1. Consider the nonlinear system $\dot{x}(t) = A\phi(x(t))$ where $A \in \mathbb{R}^{n \times n}$ and $\phi(\cdot) \ : \ \mathbb{R}^n \to \mathbb{R}^n$ is such that

$$\phi(0) = 0 \ , \quad \phi(x) = [\phi_1(x_1), \cdots, \phi_n(x_n)]'$$

Show that the equilibrium point $x = 0$ is GAS wherever there exists a diagonal positive matrix P such that $A'P + PA < 0$. To this end, make use of the so called Persidiskii-Lyapunov function

$$v(x) = \sum_{i=1}^{n} P_{ii} \int_0^{x_i} \phi_i(\xi) d\xi$$

and verify under which conditions it is positive definite and radially unbounded.

Problems

2. Consider a Lur'e system with

$$H(s) = \frac{s+10}{(s+1)(s^2+2s+2)}$$

and determine:

- The maximum value of $\kappa > 0$ such that stability is preserved with $\phi(z)$ linear.
- The maximum value of $\kappa > 0$ such that stability is preserved with $\phi(z)$ nonlinear, provided by a quadratic Lyapunov function.
- The maximum value of $\kappa > 0$ such that stability is preserved with $\phi(z)$ nonlinear, provided by the Popov-Lyapunov function.
- The maximum value of $\kappa > 0$ using H_∞ theory.

Problems

3. Consider a linear plant with transfer function $H(s)$ plus a linear feedback $w(t) = -\delta z(t)$ where $\delta \in \mathbb{R}$. For $H(s)$ given by

$$H(s) = \frac{(s-1)(s-2)}{(s+1)^2(s+4)}$$

determine the upper bound δ_{max} such that asymptotic stability is preserved for all $0 \leq \delta \leq \delta_{max}$ using :

- The H_∞ theory
- The Popov criterion
- The Routh criterion

Compare the obtained results from the root locus plot (with respect to δ) of the closed-loop system.

Problems

4. Consider a MIMO Lur'e system such that

$$H(s) = C(sI - A)^{-1}B \in \mathbb{C}^{m \times m}$$

   Determine :

   - the stability conditions provided by a quadratic Lyapunov function.
   - the stability conditions provided by a Popov-Lyapunov type function.

5. Consider a SISO Lur'e system. Generalize the Popov criterion to cope with nonlinear functions belonging to the sector $(-\kappa, \kappa)$.