Continuous-Time Switched Dynamical Systems

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1. CHAPTER I - Introduction and Preliminaries
   - LTI Systems
   - Parseval's Theorem
   - Stability
   - $\mathcal{H}_2$ Norm
   - $\mathcal{H}_\infty$ Norm
   - Linear Matrix Inequalities
   - Problems

Note to the reader

- This text is based on the following main references :
  - D. Liberzon, *Switching in Systems and Control*, Birkhäuser, 2003.
  - Z. Sun, and S. S. Ge, *Switched Linear Systems : Control and Design*, Springer, London, 2005.
  - S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.
  - J. C. Geromel e R. H. Korogui, "Controle Linear de Sistemas Dinâmicos : Teoria, Ensaios Práticos e Exercícios", Edgard Blucher Ltda, 2011.

Introduction

The switched dynamical system of interest presents the following state space realization

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)} + B_{\sigma(t)}u(t) + H_{\sigma(t)}w(t)$$
$$z(t) = E_{\sigma(t)}x(t) + F_{\sigma(t)}u(t) + G_{\sigma(t)}w(t)$$

where :

- $x \in \mathbb{R}^{n_x}$ is the state
- $w \in \mathbb{R}^{n_w}$ is the perturbation
- $u \in \mathbb{R}^{n_u}$ is the control input
- $z \in \mathbb{R}^{n_w}$ is the controlled output
- $\sigma(\cdot) \ : \ t \geq 0 \to \mathbb{K} := \{1, \cdots, N\}$ is the switching function that selects one of the N available subsystems at each instant of time.
- b_σ is the affine term

Introduction

- If $b_i = 0, \ \forall i \in \mathbb{K}$, the switched system is called linear and has the origin $x = 0$ as the unique equilibrium point.
- If $b_i \neq 0$ for at least one $i \in \mathbb{K}$ the switched system is called affine and has several equilibrium points composing a region in the state space.

Switched systems can model several real world dynamical systems or can appear only in the controller structure. Indeed switched controllers can be designed to preserve stability and enhance performance of nonswitched plants overcoming other control strategies available in the literature.

Introduction

- Classes of switching functions :
  - Perturbation : $\sigma(t) \ : \ t \geq 0 \rightarrow \mathbb{K}$ is a trajectory with dwell time $T > 0$

    $$\mathcal{D}_T = \{\sigma(\cdot) \ : \ t_{k+1} - t_k \geq T \ \forall k \in \mathbb{N}\}$$

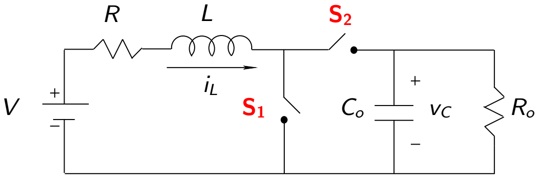
    where t_k and t_{k+1} are two successive instants of time. Notice that for $T \rightarrow \infty$ the function is constant and for $T \rightarrow 0^+$ it is arbitrary.
  - Control : The switching function is of the form $\sigma(x) \ : \ \mathbb{R}^{n_x} \rightarrow \mathbb{K}$ that must be determined in order to preserve stability and assure good performance for the closed-loop system.
- Our interest is to study the second class. For this case, the literature presents some sufficient conditions based on different types of Lyapunov functions.

Introduction

Some very simple examples are :

- Mechanical Engineering : Automatic gear car

- Electrical Engineering : DC-DC Boost converter.



Introduction

The control design of a switching function is very important not only for practical, but also for theoretical reasons. Indeed considering a simpler switched system

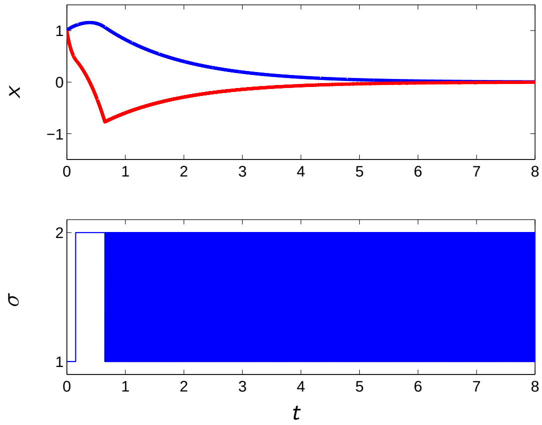
$$\dot{x}(t) = A_{\sigma}x(t)$$

we can observe the following interesting characteristics :

- If all subsystems are unstable a suitable switching function can assure stability for the overall system.
- If all subsystems are stable a suitable switching function can enhance performance compared to those of all isolated subsystems. In this case, the switching function is said to be strictly consistent. However, an inadequate switching function can take the system to instability.

Introduction

For $N = 2$ unstable subsystems evolving from $x_0 = [1\ 1]'$:



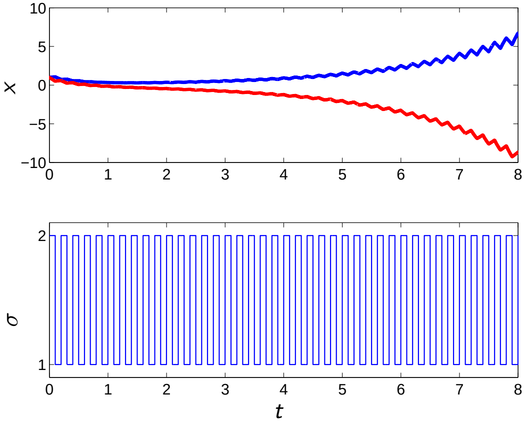
$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$$

The adopted switching function is stabilizing !

Introduction

For $N = 2$ stable subsystems evolving from $x_0 = [1\ 1]'$:



$$A_1 = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix}$$

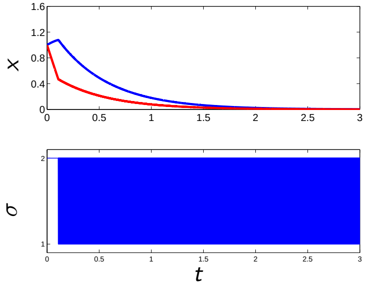
The adopted switching function is not stabilizing !

Introduction

However for the more general system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + H_{\sigma(t)}w(t)$$
$$z(t) = E_{\sigma(t)}x(t)$$

with the previously defined stable matrices A_1, A_2 and $E_1 = E_2 = I$, $H_1 = H_2 = [1 \;\; 1]'$ we have

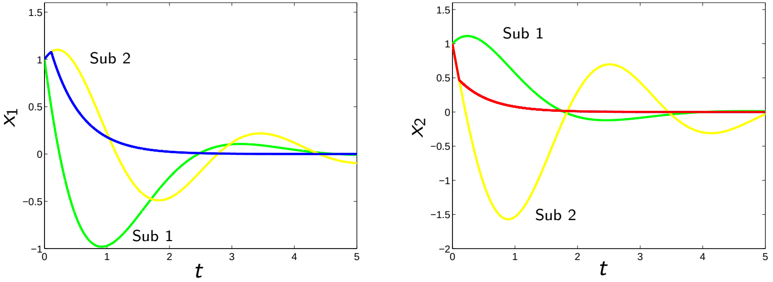


The switching function is responsible for an $\mathcal{H}_2$ performance gain greater than 73% with respect to that of each isolated subsystem

The adopted switching function is strictly consistent !

Introduction

- Time responses of Subsystems 1 and 2 and of the Switched system.



Introduction

The control design of a switching function is a good alternative to several engineering problems :

- Multiobjective Control : The switched control law

$$u(t) = K_{\sigma(x(t))}x(t)$$

where the gains $\{K_1, \cdots, K_N\}$ must be determined together with the switching function $\sigma(x)$, is very effective to assure stability and performance for the Linear Time Invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t) + Hw(t)$$

whenever several different and possibly conflicting criteria

$$z(t) = E_{\sigma(t)}x(t) + F_{\sigma(t)}u(t)$$

defined by the pair $(E_i, F_i), \forall i \in \mathbb{K}$, are imposed.

Introduction

- Linear Parameter Varying (LPV) Systems : The control law $u(t) = K_{\sigma(x(t))}x(t)$ can be applied to the LPV system

$$\Sigma(\lambda) := \begin{cases} \dot{x}(t) = A_{\lambda(t)}x(t) + B_{\lambda(t)}u(t) + Hw(t) \\ z(t) = C_{\sigma(t)}x + D_{\sigma(t)}u(t) \end{cases}$$

with

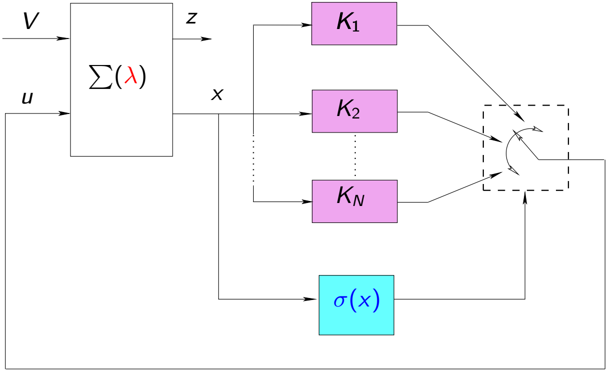
$$(A_{\lambda(t)}, B_{\lambda(t)}) = \sum_{j=1}^{N} \lambda_j(t)(A_j, B_j), \;\; \lambda(t) \in \Lambda$$

It is a good alternative to the gain scheduling control $u(t) = K_{\lambda(t)}x(t)$ whenever the uncertain parameter $\lambda \in \Lambda$ is not available. The set Λ denotes the unitary simplex defined as

$$\Lambda = \left\{ \lambda \in \mathbb{R}^N \; : \; \lambda_i \geq 0, \; \sum_{i \in \mathbb{K}} \lambda_i = 1 \right\}$$

Introduction

The switched control scheme is presented as follows

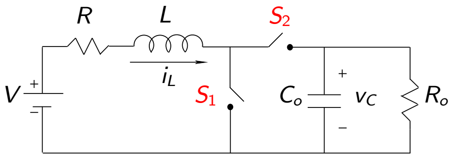


$$u(t) = K_{\sigma(x(t))}x(t)$$

Introduction

- Power Electronics : In the Boost converter presented bellow the switches S_1 and S_2 operates complementarily defining for $x = [i_L \;\; v_C]'$ the switched affine system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)}, \;\; x(0) = x_0$$

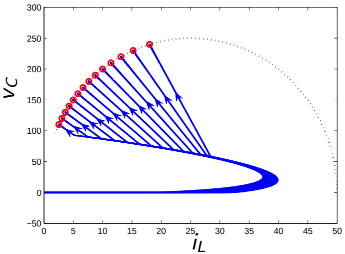


$$A_1 = \begin{bmatrix} -R/L & 0 \\ 0 & -1/R_oC_o \end{bmatrix}, \; A_2 = \begin{bmatrix} -R/L & -1/L \\ 1/C_o & -1/R_oC_o \end{bmatrix},$$

$$b_1 = b_2 = \begin{bmatrix} V/L \\ 0 \end{bmatrix}$$

Introduction

The switching function $\sigma(x)$ must be designed in order to govern the system trajectories to the desired equilibrium point $x_e \in X_e$ with $X_e \subset \mathbb{R}^{n_x}$. The phase portrait bellow shows the equilibrium points in red and the state trajectories evolving from the origin under the action of the switching function.



The design of a suitable switching rule may be more effective than the PWM technique very common in the literature!

Introduction

- Networked control : Considering that the control input is transmitted through a limited bandwidth channel, whose sampling periods $T_i, \ i \in \mathbb{K}$, are defined by the user respecting the minimum value allowed for transmission $T_i > T_\star > 0$, we have the system

$$\dot{x}(t) = Ax(t) + Bu_k(t) + Hw(t), \ x(0) = 0$$
$$z(t) = Ex(t) + Fu_k(t)$$

where the control input is a piecewise constant signal

$$u_k(t) = u(t_k) = u[k], \ \forall t \in [t_k, t_{k+1})$$

The interval between two successive instants of time is $t_{k+1} - t_k \in \mathcal{T}, \ \forall k \in \mathbb{N}$, with $\mathcal{T} = \{T_i, \ i \in \mathbb{K}\}$.

Introduction

We can define a self-triggered control problem that consists in selecting a suitable sampling period T_i, $i \in \mathbb{K}$, at each interval of time. For each sampling period we can define a discrete-time switched **equivalent** system

$$x[k+1] = A_{d\sigma} x[k] + B_{d\sigma} u[k], \ x(0) = x_0$$
$$z[k] = E_{d\sigma} x[k] + F_{d\sigma} u[k]$$

and to design a switching function $\sigma(x)$ taking into account two possibly conflicting criteria, as for instance, the $\mathcal{H}_2$ performance, which generally induces small values for the sampling period, and the limited bandwidth, which constraints this behavior.

LTI Systems

- Consider a continuous-time LTI system given by

$$\dot{x}(t) = Ax(t) + Hw(t), \;\; x(0) = x_0$$
$$z(t) = Ex(t) + Gw(t)$$

where $x \in \mathbb{R}^{n_x}$ is the state, $w \in \mathbb{R}^{n_x}$ is the perturbation and $z \in \mathbb{R}^{n_z}$ is the controlled output. The general solution of the dynamic equation is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Hw(\tau)d\tau, \;\; \forall t \geq 0$$

For $w(t) = w_0\delta(t)$ with $\delta(\cdot)$ the impulse function, we have

$$x(t) = e^{At}(x_0 + Hw_0), \;\; \forall t \geq 0$$

LTI Systems

- Using the general solution, the controlled output is

$$
\begin{aligned}
z(t) &= Ee^{At}x_0 + \int_0^t Ee^{A(t-\tau)}Hw(\tau)d\tau + Gw(t) \\
&= \underbrace{Ee^{At}x_0}_{h_{wz0}(t)} + \int_0^t \underbrace{\left(Ee^{A(t-\tau)}H + G\delta(t-\tau)\right)}_{h_{wz}(t-\tau)} w(\tau)d\tau \\
&= h_{wz0}(t) + h_{wz}(t) * w(t)
\end{aligned}
$$

where

- $h_{wz0}(t)$ depends only on the initial condition without any input.
- $h_{wz}(t) = Ee^{At}H + G\delta(t)$ is the impulse response for zero initial condition.

LTI Systems

- We can characterize the system in the frequency domain by applying the Laplace Transform to the state space equations

$$\hat{z}(s) = H_{wz0}(s) + H_{wz}(s)\hat{w}(s)$$

with

$$\begin{aligned} H_{wz0}(s) &= \mathcal{L}\{h_{wz0}(t)\} = E(sI - A)^{-1}x_0 \\ H_{wz}(s) &= \mathcal{L}\{h_{wz}(t)\} = E(sI - A)^{-1}H + G \end{aligned}$$

$$\Downarrow$$

$H_{wz}(s)$ is the transfer function of the system

- For SISO systems we can write $H_{wz}(s) = N(s)/D(s)$ with $N(s)$ and $D(s)$ being polynomials.

LTI Systems

- The poles of $H_{wz}(s)$ are the roots of $D(s) = \det(sI - A) = 0$.
- The zeros of $H_{wz}(s)$ are the roots of

$$
\begin{aligned}
N(s) &= \det(sI - A) H_{wz}(s) \\
&= \det\left(\begin{bmatrix} sI - A & -H \\ 0 & H_{wz}(s) \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} I & 0 \\ -E(sI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} sI - A & -H \\ E & G \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} sI - A & -H \\ E & G \end{bmatrix} \right) \\
&= 0
\end{aligned}
$$

Notice that $H_{wz0}(s)$ and $H_{wz}(s)$ have the same denominators !

LTI Systems

- The frequency response of a system with transfer function $H_{wz}(s)$ is simply given

$$H_{wz}(j\omega), \ \forall \omega \in \mathbb{R}$$

which imposes that $j\omega \in \mathcal{D}\{H_{wz}\}$. This means that the imaginary axis must belong to the domain of $H_{wz}(s)$ and, consequently, all poles must be located in the region $\mathrm{Re}(s) < 0$.

- The equality

$$H_{wz}(j\omega) = \int_0^\infty h_{wz}(t) e^{-j\omega t} dt$$

holds and provides the Fourier transform of $h_{wz}(t)$.

Norms

- Consider a vector $x \in \mathbb{C}^{n_x}$ and denote $x^\sim$ its conjugate transpose. The quantity

$$\|x\| := \sqrt{x^\sim x} = \sqrt{\sum_{i=1}^{n_x} |x_i|^2}$$

is the Euclidean norm of the vector x.

- Consider a trajectory $x(t) \in \mathbb{C}^{n_x}$ defined for all $t \geq 0$. The quantity

$$\|x\|_2 := \sqrt{\int_0^\infty \|x(t)\|^2 dt} = \sqrt{\int_0^\infty x(t)^\sim x(t) dt}$$

is the $\mathcal{L}_2$ norm of the trajectory $x(t)$.

Preliminaries

An important result that relates the integral of a trajectory $x(t) \in \mathbb{R}^{n_x}$ defined for all $t \geq 0$ with the integral of its Fourier transform is the Parseval's Theorem.

Parseval's Theorem

Consider a function $x(t) \in \mathbb{R}^{n_x}$ and its Laplace transform $\hat{x}(s) \in \mathbb{C}^{n_x}$ such that $0 \in \mathcal{D}\{\hat{x}(s)\}$, then the following equality

$$\|x(t)\|_2^2 = \frac{1}{\pi} \int_0^{\infty} \|\hat{x}(j\omega)\|^2 d\omega$$

is verified.

The proof is based on the inverse Laplace transform applied with Γ being the imaginary axis, that is

$$x(t) = \frac{1}{2\pi j} \int_\Gamma \hat{x}(s) e^{st} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{x}(j\omega) e^{j\omega t} d\omega$$

Preliminaries

- From the norm definition we have

$$\begin{aligned}
\|x\|_2^2 &= \int_0^\infty x(t)^\sim x(t)dt \\
&= \int_0^\infty x(t)^\sim \left(\frac{1}{2\pi} \int_{-\infty}^\infty \hat{x}(j\omega)e^{j\omega t} d\omega \right) dt \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_0^\infty x(t)' e^{-j\omega t} dt \right)^* \hat{x}(j\omega) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \hat{x}(j\omega)^\sim \hat{x}(j\omega) d\omega \\
&= \frac{1}{\pi} \int_0^\infty \|\hat{x}(j\omega)\|^2 d\omega
\end{aligned}$$

where the last equality is a consequence of $\hat{x}(j\omega)^* = \hat{x}(-j\omega)$ because $x(t)$ is real.

Stability

- In order to study stability, let us consider the simpler LTI system

$$\dot{x} = Ax, \ \ x(0) = x_0$$

which has an equilibrium point at the origin, which is the unique, whenever $\det(A) \neq 0$. The solution of this system is

$$x(t) = e^{At}x_0$$

where the exponential calculation is given by

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

If all eigenvalues of matrix A have negative real part, the system is globally asymptotically stable (Hurwitz stable).

Stability

- The asymptotic stability can also be investigated by using the Lyapunov method. Let us consider the quadratic Lyapunov function candidate

$$v(x) = x'Px, \;\; P = P' > 0 \in \mathbb{R}^{n_x \times n_x}$$

which defines a distance of the trajectory x to the origin $x = 0$. The time derivative of $v(x)$ along the trajectories of the linear system provides

$$\dot{v}(x) = \dot{x}'Px + x'P\dot{x} = x'(A'P + PA)x = -x'Qx$$

where Q is a symmetric matrix given by

$$A'P + PA = -Q$$

If Q is positive definite then $\dot{v}(x(t)) < 0, \;\forall x \neq 0$, and we can conclude that the origin is globally asymptotically stable.

Stability

- The celebrated Lyapunov Theorem is stated as follows :

Lyapunov Theorem

Matrix A is Hurwitz stable **if and only if** for any given $Q > 0$ there exists a positive definite symmetric matrix P satisfying the Lyapunov equation

$$A'P + PA + Q = 0$$

Moreover, matrix P is the unique solution of this equation.

- Sufficiency : Follows from the already mentioned fact that if the Lyapunov equation with $Q > 0$ has a solution $P > 0$ then $\dot{v}(x(t)) < 0$, $\lim_{t \to \infty} x(t) = 0$ and, consequently, the system is globally asymptotically stable.

Stability

- Necessity : We need to show that, if the system is globally asymptotically stable, then the Lyapunov equation has a unique solution. Consider matrix P defined by

$$P = \int_0^\infty e^{A't} Q e^{At} dt$$

a possible solution of the Lyapunov equation. This integral always exists since $\mathrm{Re}(\lambda_i(A)) < 0, \ i \in \mathbb{K}$. Moreover, multiplying to the right by an arbitrary vector $0 \neq \chi \in \mathbb{R}^{n_x}$ and to the left by the transpose, we have

$$\chi' P \chi = \int_0^\infty x' Q x dt$$

with $x(t) = e^{At}\chi$. We can conclude that P is symmetric and positive definite since $Q > 0$.

Stability

- Now substituting the solution P at the equation, we have

$$\begin{aligned} A'P + PA &= A'\left(\int_0^\infty e^{A't}Qe^{At}dt\right) + \left(\int_0^\infty e^{A't}Qe^{At}dt\right)A \\ &= \int_0^\infty \frac{d}{dt}\left(e^{A't}Qe^{At}\right)dt = e^{A't}Qe^{At}\Big|_0^\infty \\ &= \lim_{t\to\infty} e^{A't}Qe^{At} - Q \\ &= -Q \end{aligned}$$

where the last equality is a consequence of the fact that A is Hurwitz stable and, therefore

$$\lim_{t\to\infty} e^{A't}Qe^{At} = 0$$

Stability

- In order to show that it is the unique solution, suppose that there exists another one $\tilde{P} \neq P$, which provides

$$A'(P - \tilde{P}) + (P - \tilde{P})A = 0$$

Multiplying to the left by $e^{A't}$ and to the right by the transpose, we have

$$e^{A't}\left(A'(P - \tilde{P}) + (P - \tilde{P})A\right)e^{At} = \frac{d}{dt}\left(e^{A't}(P - \tilde{P})e^{At}\right) = 0$$

Hence

$$e^{A't}(P - \tilde{P})e^{At} = cte, \ \forall t \geq 0$$

Evaluating the equality for $t = 0$ and $t \to \infty$ we conclude that $P = \tilde{P}$ is the unique solution.

$\mathcal{H}_2$ Norm

- For a stable LTI system, we can define two important performance indexes : the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.
- In order to calculate both norms, let us recall the LTI system of interest

$$\dot{x}(t) = Ax(t) + Hw(t), \;\; x(0) = 0$$
$$z(t) = Ex(t) + Gw(t)$$

with the associated transfer function

$$H_{wz}(s) = E(sI - A)^{-1}H + G$$

$\mathcal{H}_2$ Norm

$\mathcal{H}_2$ norm

For strictly proper asymptotically stable LTI systems, the $\mathcal{H}_2$ norm is defined as

$$\|H_{wz}(s)\|_2 = \left(\int_0^{\infty} \mathrm{Tr}\left(h_{wz}(\tau)' h_{wz}(\tau) \right) d\tau \right)^{1/2}$$

Notice that it depends on the impulse response of the system. However, using the Parseval's Theorem, we can express it in the frequency domain as being

$$\|H_{wz}(s)\|_2^2 = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \mathrm{Tr}\left(H_{wz}(-j\omega)' H_{wz}(j\omega) \right) d\omega \right)$$

$\mathcal{H}_2$ Norm

- The impulse response of the system is

$$h_{wz}(t) = Ee^{At}H + G\delta(t)$$

and, therefore, we have

$$\|H_{wz}(s)\|_2^2 = \int_0^\infty \mathrm{Tr}\left((Ee^{At}H + G\delta(t))'(Ee^{At}H + G\delta(t))\right) dt$$

$$= \mathrm{Tr}\left(H'\left(\int_0^\infty e^{A't}E'Ee^{At}dt\right)H\right) + 2\mathrm{Tr}(H'E'G)$$

$$+ \mathrm{Tr}(G'G)\int_0^\infty \delta(t)^2 dt$$

Notice that

$$\int_0^\infty \delta(t)^2 dt = \frac{1}{\pi}\int_0^\infty d\omega \to \infty$$

$\mathcal{H}_2$ Norm

- Hence, the $\mathcal{H}_2$ norm can be calculated only for strictly proper systems, that is, for $G = 0$ and is given by

$$\|H_{wz}(s)\|_2^2 = \{\mathrm{Tr}(H'P_oH) \ : \ A'P_o + P_oA + E'E = 0\}$$

where

$$P_o = \int_0^\infty e^{A't}E'Ee^{At}dt$$

is the observability gramian.

- Alternatively, this quantity can be determined as the solution of a convex optimization problem. Indeed, notice that the solution $P > 0$ of the inequality

$$A'P + PA + E'E < 0$$

satisfies the Lyapunov equation $A'P + PA + E'E = -S$ for an arbitrary $S > 0$.

$\mathcal{H}_2$ Norm

- Hence, we have

$$P = \int_0^\infty e^{A't}(E'E + S)e^{At}dt > P_o$$

and, therefore, we have

$$\|H_{wz}(s)\|_2^2 = \inf_{P>0} \{\mathrm{Tr}(H'PH) \ : \ A'P + PA + E'E < 0\}$$

- Alternatively, using in the $\mathcal{H}_2$ norm definition and the circularity property $\mathrm{Tr}\left(h_{wz}(\tau)'h_{wz}(\tau)\right) = \mathrm{Tr}\left(h_{wz}(\tau)h_{wz}(\tau)'\right)$ we can obtain the $\mathcal{H}_2$ norm from the controllability gramian.

$$\|H_{wz}(s)\|_2^2 = \{\mathrm{Tr}(EP_cE') \ : \ AP_c + P_cA' + HH' = 0\}$$

$$\|H_{wz}(s)\|_2^2 = \inf_{P>0} \{\mathrm{Tr}(EPE') \ : \ AP + PA' + HH' < 0\}$$

$\mathcal{H}_\infty$ Norm

- The $\mathcal{H}_\infty$ norm is defined as follows.

$\mathcal{H}_\infty$ norm

For asymptotically stable LTI systems, the $\mathcal{H}_\infty$ norm is defined as

$$\|H_{wz}(s)\|_\infty = \sup_{\omega\in\mathbb{R}} \mu_{max}\{H_{wz}(j\omega)\}$$

where $\mu_{max}\{\cdot\}$ is the maximum singular value of $H_{wz}(j\omega)$

- The maximum singular value can be calculated as

$$\mu_{max}\{H_{wz}(j\omega)\} = \max_{i=1,\cdots,n_x} \sqrt{\lambda_i\{H_{wz}(j\omega)^\sim H_{wz}(j\omega)\}}$$

where $\lambda_i\{V\}$ is the i-th eigenvalue of matrix V.
- For SISO systems $\mu_{max}\{H_{wz}(j\omega)\} = |H_{wz}(j\omega)|$.
- Differently from the $\mathcal{H}_2$ case, it does not require that the system be strictly proper.

$\mathcal{H}_\infty$ Norm

- Notice that the $\mathcal{H}_\infty$ norm depends on the transfer function of the system.
- However, using the Parseval's Theorem it is possible to find a condition on the time-domain. Indeed, considering that $w(t) \in \mathcal{L}_2$ and $\hat{z}(s) = H_{wz}(s)\hat{w}(s)$ we can write

$$\begin{aligned}
\int_0^\infty z(t)'z(t) &= \frac{1}{\pi}\int_0^\infty \hat{z}(j\omega)^\sim \hat{z}(j\omega)d\omega \\
&= \frac{1}{\pi}\int_0^\infty \hat{w}(j\omega)^\sim H_{wz}(j\omega)^\sim H_{wz}(j\omega)\hat{w}(j\omega)d\omega \\
&\leq \|H_{wz}(s)\|_\infty^2 \int_0^\infty w(t)'w(t)dt
\end{aligned}$$

and, therefore, we have

$$\|H_{wz}(s)\|_\infty^2 \leq \rho \iff \|z(t)\|_2^2 \leq \rho\|w(t)\|_2^2$$

$\mathcal{H}_\infty$ Norm

- Notice that, adopting the quadratic Lyapunov function $v(x) = x'Px$, $P > 0$, and imposing

$$\dot{v}(x(t)) < -z(t)'z(t) + \rho w(t)'w(t), \ \forall t \geq 0$$

for some $\rho > 0$, after integrating both sides from $t = 0$ until $t \to \infty$ we obtain

$$\int_0^\infty \dot{v}(x(t)) < \int_0^\infty -z(t)'z(t) + \rho w(t)'w(t)dt$$

Since the system is globally asymptotically stable $v(x(\infty)) = 0$. Moreover $v(x(0)) = 0$ because $x(0) = 0$ and as a consequence

$$\int_0^\infty z(t)'z(t) - \rho w(t)'w(t)dt < 0$$

which leads us to the conclusion that $\|H_{wz}(s)\|_\infty^2 \leq \rho$.

$\mathcal{H}_\infty$ Norm

- Hence, it suffices to impose that the inequality

$$\begin{aligned}\dot{v}(x(t)) &= \left\{\dot{x}(t)'Px(t) + x(t)'P\dot{x}(t) + z(t)'z(t) - \rho w(t)'w(t)\right\} \\ &\quad - z(t)'z(t) + \rho w(t)'w(t) \\ &= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} A'P + PA + E'E & PH + E'G \\ H'P + G'E & G'G - \rho I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\ &\quad - z(t)'z(t) + \rho w(t)'w(t) \\ &< -z(t)'z(t) + \rho w(t)'w(t)\end{aligned}$$

is verified, which is true whenever

$$\begin{bmatrix} A'P + PA + E'E & PH + E'G \\ H'P + G'E & G'G - \rho I \end{bmatrix} < 0$$

$\mathcal{H}_\infty$ Norm

- The $\mathcal{H}_\infty$ norm can be calculated by the optimization problem

$$\|H_{wz}(s)\|_\infty^2 = \inf_{\{\rho>0, P>0\}} \left\{ \rho : \begin{bmatrix} A'P + PA + E'E & PH + E'G \\ H'P + G'E & G'G - \rho I \end{bmatrix} < 0 \right\}$$

or, alternatively, using duality by

$$\|H_{wz}(s)\|_\infty^2 = \inf_{\{\rho>0, P>0\}} \left\{ \rho : \begin{bmatrix} AP + PA' + HH' & PE' + HG' \\ EP + GH' & GG' - \rho I \end{bmatrix} < 0 \right\}$$

Linear Matrix Inequalities

- Linear matrix inequalities (LMIs) are essential in the analysis and control design of dynamical systems and to several optimization problems.

Linear Matrix Inequality

An LMI is expressed as

$$\mathcal{A}(x) < 0$$

with

$$\mathcal{A}(x) = A_0 + \sum_{i=1}^{n} A_i x_i$$

where $A_i \in \mathbb{R}^{m\times m}, \; i = 0, \cdots, n$ are symmetric matrices and $x_i \in \mathbb{R}$ is the i-th component of vector x.

Linear Matrix Inequalities

- Notice that $\mathcal{A}(x) : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ is a linear function of the vector $x \in \mathbb{R}$.

Convex set

The set of vectors $x \in \mathbb{R}$ satisfying the linear matrix inequality $\mathcal{A}(x) < 0$ is convex.

- Indeed, notice that for two generic points $x_a, \; x_b \in \mathbb{R}^n$ the segment between them is $x = \alpha x_a + (1-\alpha)x_b$ for $0 \leq \alpha \leq 1$. Assuming that $\mathcal{A}(x_a) < 0$ and $\mathcal{A}(x_b) < 0$, we have

$$\begin{aligned} \mathcal{A}(x) &= \mathcal{A}(\alpha x_a + (1-\alpha)x_b) \\ &= \alpha \mathcal{A}(x_a) + (1-\alpha)\mathcal{A}(x_b) \\ &< 0 \end{aligned}$$

where the second equality is due to the fact that $\mathcal{A}(x)$ is linear.

Linear Matrix Inequalities

- An important result used to linearise some nonlinear constraints is the Schur Complement.

Schur Complement

A linear matrix inequality

$$\mathcal{A}(x) = \begin{bmatrix} S(x) & V(x) \\ V(x)' & Q(x) \end{bmatrix} < 0$$

is equivalent to any of the two nonlinear inequalities

a) $S(x) < 0$ and $Q(x) - V(x)'S(x)^{-1}V(x) < 0$

b) $Q(x) < 0$ and $S(x) - V(x)Q(x)^{-1}V(x)' < 0$

Linear Matrix Inequalities

- Indeed for part a), notice that $S(x) < 0$ also implies that $S(x)^{-1} < 0$. As a consequence, matrix

$$U(x) = \begin{bmatrix} I & 0 \\ V(x)'S(x)^{-1} & I \end{bmatrix}$$

is nonsingular and allows us to write $\mathcal{A}(x) = U(x)\mathcal{B}(x)U(x)'$, where

$$\mathcal{B}(x) = \begin{bmatrix} S(x) & 0 \\ 0 & Q(x) - V(x)'S(x)^{-1}V(x) \end{bmatrix}$$

Hence matrix $\mathcal{A}(x) < 0$ if and only if $\mathcal{B}(x) < 0$. The proof of part b) is similar.

Linear Matrix Inequalities

- **Example 1 :** Convert the linear inequalities $2x_1 + 3x_2 < 7$, $-x_1 + x_2 < 5$ and $2x_1 - 4x_2 < -4$ in a matrix form.

  Answer :

$$A_0 = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Linear Matrix Inequalities

- **Example 2 :** Convert the nonlinear inequality $(x_1 - 1)^2 + 2(x_2 - 2)^2 < 5^2$, which is an ellipse with focus in (1,2), in a linear matrix inequality.

  Answer :

  Performing the Schur Complement, we have that it is equivalent to

$$\begin{bmatrix} 2(x_2 - 2)^2 - 25 & x_1 - 1 \\ x_1 - 1 & -1 \end{bmatrix} < 0$$

  performing it again, we obtain

$$\begin{bmatrix} -25 & x_1 - 1 & x_2 - 2 \\ x_1 - 1 & -1 & 0 \\ x_2 - 2 & 0 & -1/2 \end{bmatrix} < 0$$

  where matrices $A_0,\ A_1,\ A_2 \in \mathbb{R}^{3\times 3}$ can be directly determined

Linear Matrix Inequalities

- The concepts we have just presented are important to solve optimization problems described as

$$\inf_{x}\{c'x \ : \ \mathcal{A}(x) < 0\}$$

where $c \in \mathbb{R}^n$.

- In the specific context of control design, two very important problems can be written as the optimization problem just presented, to know, the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ norms of the system characterized by the transfer function

$$H_{wz}(s) = E(sI - A)^{-1}H + G$$

.

Linear Matrix Inequalities

- As already mentioned the $\mathcal{H}_2$ norm of the system $H_{wz}(s)$ can be determined through the solution of the following convex optimization problem

$$\|H_{wz}(s)\|_2^2 = \inf_{P>0} \{\mathrm{Tr}(H'PH) \ : \ A'P + PA + E'E < 0\}$$

- Notice that, this problem can be written as

$$\inf_{x} \{c'x \ : \ \mathcal{A}(x) < 0\}$$

Linear Matrix Inequalities

- Indeed, considering the decision vector $x = [x_1, \cdots, x_m]'$, we have

$$P = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{F_1} x_1 + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F_2} x_2 + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{F_3} x_3 > 0$$

and

$$A'P + PA + E'E =$$
$$= (A'F_1 + F_1 A)x_1 + (A'F_2 + F_2 A)x_2 + (A'F_3 + F_3 A)x_3 + E'E < 0$$

The objective function can be written as

$$\inf_x \underbrace{\left[\mathrm{Tr}(H'F_1 H) \quad \mathrm{Tr}(H'F_2 H) \quad \mathrm{Tr}(H'F_3 H) \right]}_{c'} x$$

Linear Matrix Inequalities

- **Example 3** : Given a system with transfer function

$$H(s) = \frac{s+2}{s^3 + 2.4s^2 + 2.8s + 0.8}$$

  - Determine the system state space realization.
  - Using the LMILAB from Matlab solve the optimization problems in order to calculate $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.
  - Compare the results with the ones obtained by the commands "normh2" and "normhinf" from Matlab.

Problems

1) Obtain the state space realization of the following systems

- $H(s) = \dfrac{s^2 + 5s + 3}{s(s^2 + 5s + 6)}$

- $H(s) = \dfrac{s^2 + 0.1s}{s^2 + 0.1s + 10}$

2) Show that for an arbitrary nonsingular matrix $T \in \mathbb{R}^{n \times n}$ the state space realization $(T^{-1}AT, T^{-1}B, CT, D)$ also represents the transfer function $H(s)$ with realization (A, B, C, D).

Problems

3) Consider the differential equation

$$\ddot{\theta} + 4\dot{\theta} + 4\theta = 0, \;\; \theta(0) = 1, \;\; \dot{\theta}(0) = 0$$

- Determine its solution θ and the output $\dot{\theta} + 2\theta$.
- Determine its state space representation.
- Determine an equivalent state space representation for null initial conditions.

4) Using Laplace transform show that for $A \in \mathbb{R}^{n \times n}$ the equality

$$(sI - A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$$

Problems

5) Consider the following continuous-time system and determine :

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w, \ \ x(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ z &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

- The response $z(t)$ to the input $w(t) = e^{-2t}$.
- The transfer function $H_{wz}(s)$, its domain and the impulse response $h_{wz}(t)$.
- The integral value $I = \int_0^\infty h_{wz}(t)\sin(t)dt$.

6) Show that for any matrix $M \in \mathbb{R}^{n_x \times n_x}$ we have

$$\mathrm{Tr}(M) = \sum_{i=1}^{n_x} \lambda_i(M), \ \ \text{and} \ \ \det(M) = \Pi_{i=1}^{n_x} \lambda_i$$

where $\lambda_i(M), \ i = 1, \cdots, n_x$ are the eigenvalues of matrix M.

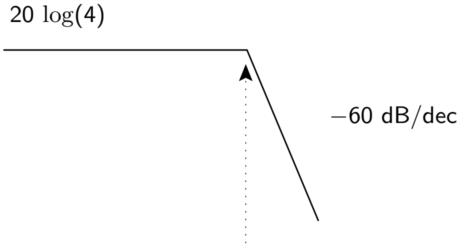
Problems

7) Using Parseval's Theorem determine the value of the integral

$$I = \int_0^\infty f(t)^2 dt$$

for $f(t) = e^{-2t}$.

8) Figure presents the Bode diagram of the minimum phase system $\hat{z} = H_{wz}(s)\hat{w}$ with real poles



- Determine the $\mathcal{H}_2$ norm.
- Determine the $\mathcal{H}_\infty$ norm.

Problems

9) Consider the following asymptotically stable transfer functions
   - $H(s) = \frac{(s+2)}{(s^2+2s+5)(s+1)}$
   - $H(s) = \frac{(s-2)}{(s^2+2s+5)(s+1)}$
   - $H(s) = \frac{(s-2)^2}{(s^2+2s+5)(s+1)}$

   Determine the $\mathcal{H}_2$ norm of each transfer function using gramians and a numerical routine of LMIsolver.

10) Consider the following asymptotically stable transfer functions
   - $H(s) = \frac{(s+2)}{(s^2+2s+5)(s+1)}$
   - $H(s) = \frac{(s-2)}{(s^2+2s+5)(s+1)}$
   - $H(s) = 1 + \frac{(s-2)^2}{(s^2+2s+5)(s+1)}$

   Determine the $\mathcal{H}_\infty$ norm of each transfer function using the singular value diagram and a numerical routine of LMIsolver.

Problems

11) Consider matrices $A \in \mathbb{R}^{n_x \times n_x}$ and $H \in \mathbb{R}^{n_x \times n_w}$. Using the Laplace transform, show that the square matrix

$$\Gamma = \begin{bmatrix} A & H \\ 0 & 0 \end{bmatrix}$$

is such that

$$e^{\Gamma t} = \begin{bmatrix} e^{At} & \int_0^t e^{At} dt H \\ 0 & I \end{bmatrix}$$

12) Show that for $T^{-1}AT = \Lambda \in \mathbb{R}^{n_x \times n_x}$ diagonal, we have

$$e^{At} = Te^{\Lambda t}T^{-1}$$

Problems

13) Consider a symmetric matrix $M = M' \in \mathbb{R}^{n_x \times n_x}$. Show that :

- All its eigenvalues and eigenvectors are real.
- All its eigenvectors are orthogonal.
- Determine $V \in \mathbb{R}^{n_x \times n_x}$ and $\Lambda \in \mathbb{R}^{n_x \times n_x}$ diagonal such that

$$V^{-1}RV = \Lambda, \; V^{-1} = V'$$

- For all $x \neq 0 \in \mathbb{M}^{n_x}$ it is true that

$$\lambda_{min} \leq \frac{x'Mx}{x'x} \leq \lambda_{max}$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of M, respectively.

Problems

14) Consider the system $\dot{x} = Ax$ and that there exist matrices $P > 0$ and $Q > 0$ such that

$$A'P + PA + Q = 0$$

Show that :

- The inequality $v(x(t)) = e^{-\alpha t} v(x(0))$ is verified and determine α.
- All eigenvalues of A are such that $\mathrm{Re}(\lambda_j(A)) < -\alpha/2$, $\forall j = 1, \cdots, n_x$ where $\alpha/2$ is the decay rate of the system.