This paper investigates through numerical simulations the limits of performance for a class of feedforward and feedback compensators used for structural acoustic isolation, where the emphasis is on controlling the structural vibration that is responsible for the sound radiation. The proposed designs aim to attenuate the sound pressure transmitted through a double panel system filled with absorption material. The controller must reduce the noise radiated through the back panel when the front panel is excited by an external force that causes structural vibration. A point moment simulating a piezoelectric patch attached to the back panel is the actuator. In the feedforward setting, the pre-filter assumes full information of the exogenous disturbance. On the other hand, the feedback designs assume full information of the state. The $H_2$ and $H_\infty$ norms are the optimality criteria used for both the filter design and the control design. All four designs are cast in the linear matrix inequality framework and incorporate parametric uncertainties described by a bounded convex polyhedral domain. It is shown that the performance of the feedforward and the feedback compensators are equivalent when model uncertainties are not taken into account. Otherwise, considering uncertainties, the feedback compensators have a more robust behavior.

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1. Introduction

One of the least desirable aspects of modern technology has been the generation of unwanted noise. The noise sources are, among others, manufacturing machinery, vehicles and fluid flows. Due to the increasing enforcement of ever more strict environmental protection laws, noise control became an attractive area of research. Some of the best examples of efforts in this field are found in the aerospace industry, where the effects of aircraft noise with respect to both passenger comfort and occupational health have long been a concern. A significant amount of research has already been carried out to improve the sound quality inside aircraft cabins. However, some cabins can still be a noisy environment, especially with respect to low frequency sounds [1,2]. A main source of noise inside an aircraft cabin is the so-called structural noise, generated by the vibration of the surrounding structure. Traditionally, its attenuation is done by passive methods that use absorbers to increase the sound transmission loss (TL). Basically, the absorbers consist of a double panel system with an acoustic insulation of porous materials [3].

Since the acoustic wavelength at low frequency becomes large compared to the thickness of a typical acoustic absorber (for example, a sound wave of frequency 100 Hz traveling in air under normal ambient conditions has a...
wavelength of about 3.4 m), it is in general difficult to attenuate the sound being transmitted from one space to another unless the intervening barrier is very heavy [4]. Hence, porous materials can only provide appreciable benefit at high frequencies, and, consequently, low frequency quieting by passive methods alone is in general too expensive and tends to increase significantly the weight of the aircraft. With physical space and weight being important restrictions for the aerospace industry, the practical application of passive methods for acoustic isolation still remains an open challenge.

On the other hand, with the advances of modern digital signal processing devices, active control methods have emerged as a potential alternative to passive methods for noise attenuation [5–9]. It is likely that control techniques can offer a lightweight and cost-effective method for reducing low-frequency noise [10]. Most of the conventional active noise control (ANC) techniques use microphones as acoustic sensors and loudspeakers as actuators [11]. In [12], the authors describe a methodology known as active structural acoustic control (ASAC) that uses structural actuators and acoustic sensors to increase the TL at low frequency. The ASAC methodology can be employed with two different cost functions, one is pressure and the other one is sound intensity [13], known as active sound intensity control (ASIC). In a broad sense, the above control designs fall into one of the following categories: feedback or feedforward strategy.

Among active noise control engineers, there is a common belief that feedforward controllers implemented as adaptive digital filters provide superior performance than the counterpart feedback control strategies [14]. This fact is certainly motivated by a large number of experimental results available in the literature that validate many feedforward implementations. On the other hand, this seems not to be the case for feedback control designs, where much less experimental results seem to have been published. An even more intriguing aspect is that one cannot assure that feedback control cannot provide similar performance, since there is not yet an oriented study from the point of view of the control theory to confirm this claim. Previous works along these lines are [15,16] that explored the relationship between these two control paradigms, and [17] that further investigated the achievable performance by means of the classical bode integral constraints. There are some comparative results available, like in [18], where the authors provide an analytical comparison of active control methods using modal acoustic transfer impedance and mobility matrices, and in [19], where the authors use an active power-flow-based approach for structural vibration control, but they do not exactly address the same questions we focus here. The investigation of the limits of performance that can be achieved through feedback and feedforward controllers in ANC applications remains an open field of research.

Our paper describes some new results along these lines. We investigate the limits of performance for a class of feedforward and feedback compensators used for structural acoustic isolation. We shall emphasize that the feedforward controllers designed in this article are not adaptive. The aim of the controller is to suppress the structural vibration that is responsible for the sound pressure transmitted through a double panel system. The proposed configuration is presented in Section 2, where its dynamics is derived using the finite element method. Basically, the configuration consists of a double panel system with an acoustic insulation of porous material inside. The control input \( u \) is designed to reduce the noise radiated through the back panel when the front panel is excited by an external force \( w \) that causes structural vibration. In the real-life situation, the excitation would typically be a diffuse sound field, but here a concentrated force is used for simplicity. For this model, we compute a set of numerical frequency response functions (FRFs) from the control input \( u \) and from the unknown disturbance \( w \) to the array of simulated pressures.

The proposed control strategies are presented in Section 3 for a continuous-time model. We introduce in Section 3.1 the \( H_2 \) and \( H_{\infty} \) norms as the optimality criteria used for both the feedforward and the feedback designs. Based on these norms, we formulate the proposed control problem as a convex programming problem described by linear matrix inequalities (LMIs). We show in Section 3.2 that the proposed feedforward design can be regarded as a pre-filtering problem. Next, in Section 3.3, the state-feedback control design is presented. Since this work focuses on comparing the performance of both feedback and feedforward designs, we assume that the pre-filter design has full information of the exogenous disturbance, and that the feedback design has full information of the state. These assumptions are indeed not realistic, but are suitable for our purpose, since they serve as a measure for the best achievable limits of performance. Section 4 shows that under these ideal assumptions the proposed class of feedforward and feedback strategies have identical \( H_2 \) and \( H_{\infty} \) performances. In Section 3.6, we show that with little modification all the proposed designs can incorporate parametric uncertainties described by a bounded convex polyhedral domain, since they are convex and described by linear matrix inequalities. The results for the robust controllers are presented in Section 4.4, where the feedback controllers are shown to provide a more robust behavior.

11. Notation

The notation used throughout the paper is as follows. Uppercase letters denote matrices, while lowercase letters denote vectors. The superscripts \( X^T \), \( X^* \) and \( X^{-1} \) mean, respectively, the transpose, the complex conjugate transpose, and the inverse of a matrix \( X \). The trace of a matrix \( X \) is denoted by \( \text{Tr}(X) \). The notation \( \sigma(X) \) means the largest singular value of \( X \). For symmetric matrices, \( X \succ 0 \) indicates that \( X \) is positive definite. The real part of a complex number \( z \) is denoted by \( \text{Re}(z) \). The symbols \( \eta \) and \( \mu \), respectively, denote upper bounds on the \( H_{\infty} \) and \( H_2 \) norms. We use capital letter \( T \) to denote the closed-loop transfer function and capital letter \( G \) to denote the open-loop plant (with the control law \( u(t) = 0 \)).
2. Double panel configuration

In the proposed configuration, illustrated in Fig. 1, the double panel system is composed of two thin aluminum plates filled with glass wool for acoustic insulation. We have assumed the following properties for the material:

- (Plates) Young’s modulus $E = 71 \times 10^9$ Pa, density $\rho = 2700$ kg/m$^3$, loss factor $LF = 0.01$, and thickness $h = 0.0012$ m.
- (Glass wool) $E = 150 \times 10^3$ Pa, $\rho = 10$ kg/m$^3$, $\eta = 0.1$, resistivity $\sigma = 2 \times 10^4$ Ns/m$^4$, porosity $\phi = 0.96$, tortuosity $\kappa_w = 1.2$, viscous characteristic length (c.l.) $A = 100$ mm, thermal c.l. $A' = 200$ mm.

The disturbance $w$ is a force located at the outer plate (front panel) that induces vibrational and acoustic fields across the panel. A concentrated moment, providing the control input $u$, simulates a piezoelectric actuator placed at the inner plate (back panel). A more accurate model to represent the piezoelectric patch would consist of two moments spaced by the patch length [20]. Nevertheless, for the theoretical investigation made here, a single moment is used, as this does not change the comparison of the results obtained with different controllers or the conclusions withdrawn. An array of 31 locations in the acoustic field facing the back panel simulates an array of equally spaced microphones measuring the acoustic pressure located 50 cm away (far field) from the inner plate. In active sound control, it is usual to use the microphones located at the far field for performance evaluation (the region where the sound must be attenuated). The acoustic far field is usually defined as half a wavelength from the sound source, so that this assumption would be strictly valid only for frequencies above 342 Hz for air at 20 °C.

We now present how the dynamics of this system was derived. The outer plate (front panel) is modeled using 1D degenerated Reissner–Mindlin plate elements [21] that take into account the bending and the transverse shear. The poroelastic material is modeled using 2D linear elements based on the mixed $(U, P)$ formulation [22]. This formulation follows Biot’s theory of poroelasticity, where $U$ accounts for the skeleton displacements and $P$ for the pressure in the pore of the material. The inner plate (back panel) is modeled as an homogeneous structural panel with the same elements used for the outer plate. The outer and the inner plates are clamped to rigid supports at both ends, while the poroelastic layer is clamped at the supports and free to slide at the interface with both panels. To derive the frequency response function from a given input–output pair, numerical computations are carried out for each frequency, since the model is frequency dependent (see [23] for details). The sound pressures are obtained in the frequency domain using the boundary element method for uncoupled radiation from the plate, as proposed in [24].

Since this paper focuses at a more theoretical point of view, in the sense of comparing different control techniques rather than experimental/practical implementations, we take the set of (simulated) microphones available for performance evaluation to be the array of 31 microphones located at the far field. We shall call this array the far-field array and denote each one of these microphones by mic$_1$, mic$_2$, ..., mic$_{31}$. Letting the output be the microphones in the far-field array, we can denote by $G_{w}(j\omega)$ the frequency response function from the disturbance input $w$ to the output $z$, and by $G_{u}(j\omega)$ the FRF from the control input $u$ to $z$. The pressure $z$ is thus given by $z = G_{w}(j\omega)w + G_{u}(j\omega)u$. Fig. 2 shows the numerically computed FRFs, as described in the last paragraph, for the frequency range from DC to 500 Hz. However, it is not practical for control purposes to fit a model over the entire frequency range. Furthermore, in noise control problems, passive treatments using acoustic materials are quite efficient above a few hundreds of Hertz, while active noise control is generally limited to frequencies ranging from tens of Hertz up to a few hundreds of Hertz. Thus, in Section 4, an 8th order model will be fitted over the 80–310 Hz frequency range, containing the dominant modes of the system. Since we made 31 channels available, corresponding to each one of the 31 microphones, the FRFs $G_{w}(j\omega)$ and $G_{u}(j\omega)$ have 31 outputs. These channels, or part of them, provide the regulated output used for performance evaluation.

![Fig. 1. Double panel system (units in centimeters).](image-url)
3. Proposed control strategies

We shall present two distinct control strategies. In the first approach, we design feedforward control laws that assume full information of the disturbance input \( w \). On the other hand, the second control strategy provides feedback control laws that assume full information of the state of the model. The proposed designs provide a measure of the best achievable performance for a class of feedforward and feedback compensators. We start this section by presenting the optimality criteria used in the derivation of the control laws.

3.1. The \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) performance criteria

Among the possible performance criteria available, we are interested in the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) system norms. These are important MIMO system norms that are readily computed from a state-space realization using linear matrix inequalities. These criteria will be used for both filtering and control designs and, thus, we briefly present their derivation in this section. These are standard results found in the literature. For instance, a complete account of this topic can be found in [25–27]. In particular, a survey on control problems posed using LMIs is presented in [28].

The \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms of a complex matrix valued (or scalar valued) function \( T(s) \) are, respectively, defined as

\[
\|T(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr}[T^*(jw)T(jw)] \, dw \quad \text{and} \quad \|T(s)\|_\infty = \sup_{w \in \mathbb{R}} |T(jw)|.}
\]

(1)

Computing the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms using these frequency-based notions is not computationally tractable. In order to find suitable formulas for these norms, let us consider the following linear time-invariant minimal state-space realization:

\[
\dot{x} = \bar{A}x + \bar{B}w,
\]

\[
z = \bar{C}x + \bar{D}w,
\]

(2)

where \( x \) is the state of the system, the input \( w(t) \) is the exogenous disturbance, and \( z(t) \) is the regulated (performance) output. All system matrices have compatible dimensions. The transfer function \( T_{wd}(s) \) from the input \( w \) to the output \( z \) is thus given by

\[
T_{wd}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}.
\]

(3)

To have a finite \( \mathcal{H}_2 \) norm, system (2)–(3) needs to be strictly proper. This is not actually the case for the \( \mathcal{H}_\infty \) norm. Nevertheless, we assume that \( \bar{D} = 0 \) for all the designs, since the dynamics of the structure (the double panel system) is inherently strictly proper and we do not use weighting functions.

Lemma 3.1 and Corollary 3.3 show how one can compute the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms through a convex optimization problem described by linear matrix inequalities, for which efficient numerical semidefinite programming (SDP) solvers are available in the literature ([29] and references therein).
Lemma 3.1 (The $\mathcal{H}_2$ performance). Consider system (2)–(3). Assume that matrix $\hat{A}$ is asymptotically stable. Then the following statements are equivalent:

(i) $\|T_{wz}(s)\|_2^2 = \mu$.
(ii) There exist symmetric matrices $W > 0$ and $P > 0$ such that

$$\|T_{wz}(s)\|_2^2 = \min \mu \text{ s.t. } \text{Tr}[W]<\mu, \quad \begin{bmatrix} P & C \\ C & W \end{bmatrix} > 0, \quad \begin{bmatrix} PA + \hat{A}P & PB \\ B^TP & -I \end{bmatrix} < 0.$$  \hfill (4)

Proof. See [27, Section 4.6]. □

One can interpret the $\mathcal{H}_2$ norm of a system as the RMS value of the output when the inputs are driven by independent white noises.

Lemma 3.2 (Bounds on the $\mathcal{H}_\infty$ performance). Consider system (2)–(3). Assume that matrix $\hat{A}$ is asymptotically stable. Then the following statements are equivalent:

(i) $\|T_{wz}(s)\|_\infty < \eta$.
(ii) There exist a scalar $\eta > 0$ and a symmetric matrix $P > 0$ such that

$$\begin{bmatrix} PA + \hat{A}P & PB \\ B^TP & -I \end{bmatrix} + \begin{bmatrix} 0 & \eta I \\ \eta I & 0 \end{bmatrix} < 0.$$  \hfill (5)

Proof. See [27, Theorem 4.6.3]. □

It is clear from Lemma 3.2 that the $\mathcal{H}_\infty$ norm can be obtained through a convex optimization problem posed as linear matrix inequalities.

Corollary 3.3 (The $\mathcal{H}_\infty$ performance). Consider the assumptions in Lemma 3.2. Then

$$\|T_{wz}(s)\|_\infty = \min \eta \text{ s.t. } P > 0 \text{ and LMI (5).}$$

Proof. Follows directly from Lemma 3.2. □

There is an important interpretation for the $\mathcal{H}_\infty$ norm. Let $L_2$ denote the space of all signals with bounded finite energy. Suppose that $T_{wz}(s)$ is a proper and real stable transfer function and that the disturbance $w$ belongs to $L_2$, then the output $z$ also belongs to $L_2$ and $\|T_{wz}(s)\|_\infty = \sup \{\|z\|_2 : \|w\|_2 \leq 1\}$. Thus, the $\mathcal{H}_\infty$ norm gives a measure of the system gain for the worst-case of disturbance $w \in L_2$ having norm $\|w\|_2 = 1$.

3.2. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ feedforward designs

The basic block diagram used in this paper for the feedforward design is

\[ \begin{array}{c}
  w \\
  \downarrow \\
  F \\
  \downarrow \\
  u \\
  \downarrow \\
  G_{uw} \\
  \downarrow \\
  z \\
  \end{array} \]

where $u$ is the control law given by $u(s) = F(s)w(s)$, with $F(s)$ the compensator to be designed. Using the compensator $F(s)$, the closed-loop system $T_{wz}(s)$ from the disturbance input $w(s)$ to the output $z(s)$ is given by $T_{wz}(s) = G_{uw}(s)F(s) + G_w(s)$. The aim of the controller is to minimize the output $z$ for some class of disturbance $w$. This feedforward control problem can be equivalently regarded as a pre-filtering problem.

Let us consider that the plant model consisting of $G_u(s)$ and $G_w(s)$ has the state-space realizations given by

$$\begin{align*}
\dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \\
z(t) &= Cx(t),
\end{align*}  \hfill (6)$$

and the pre-filter $F(s)$ has the state-space realizations given by

$$\begin{align*}
\dot{x}_f(t) &= A_f x_f(t) + B_f w(t), \\
u(t) &= C_f x_f(t) + D_f w(t),
\end{align*}  \hfill (7)$$
where \( x \in \mathbb{R}^n \) is the state of the system, \( x_f \in \mathbb{R}^n \) is the state of the filter, \( w(t) \in \mathbb{R}^m \) is the exogenous disturbance, the input \( u(t) \in \mathbb{R}^p \) is the control input, and \( z(t) \in \mathbb{R}^r \) is the regulated output for performance evaluation. All system matrices have compatible dimensions. The filter has the same order as the plant. Connecting the filter to the system, we obtain the closed-loop transfer function \( T_{wz}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} \) with matrices \( \tilde{A}, \tilde{B}, \) and \( \tilde{C} \) given by

\[
\tilde{A} = \begin{bmatrix} A & B_uC_f \\ 0 & A_f \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_w + B_uD_f \\ B_f \end{bmatrix}, \quad \tilde{C} = [C 0].
\]

With the closed-loop system defined as above, we are in position to state our convex filtering problem.

**Theorem 3.4** (Convex \( H_\infty \) filtering problem). Let the plant model be described by (6), and consider the filter (7). Then, the following statements are equivalent:

1. There exists a stable dynamic filter of order \( n \) such that
   \[
   \eta^* = \min_{\tilde{P}} \| T_{wz}(s) \|_{\infty}^2.
   \]
2. The following LMI problem has a solution in the unknowns \( \eta, Y, Z, G, F, Q, D_f \):
   \[
   \eta^* = \min \eta, \quad \begin{bmatrix} Y & Z \\ Z & Z \end{bmatrix} > 0, \quad \begin{bmatrix} YC' \\ ZC' \\ CY & CZ & 0 & -\eta I \end{bmatrix} < 0,
   \]
   with the partition \( \mathbb{H}_2 \) given by
   \[
   \mathbb{H}_2 := \begin{bmatrix} AY + YA' + B_uF + F'B_u' & AZ + YA' + F'B_u' + Q' & B_w + B_uD_f \\ ZA' + YA + B_uF + Q & AZ + ZA' & B_w + B_uD_f \\ B_w + D_f'B_u' & B_w + D_f'B_u' + G' & -I \end{bmatrix}.
   \]
   Once this problem is solved, the filter parameters \( A_f, B_f \) and \( C_f \) are, respectively, given by
   \[ A_f = -(I - ZY^{-1})^{-1}QY^{-1}, \quad B_f = (I - ZY^{-1})^{-1}G, \quad C_f = -FY^{-1}. \]

**Proof.** The arguments used to prove this theorem are similar to those used in [30–32]. However, it is important to note that we cannot apply directly the results in [30], since we solve a slightly different filtering problem. We design a pre-filter \( F(s) \), and the matrix inequality (5) describing the \( H_\infty \) performance also differs.

For fixed closed-loop system matrices \( \tilde{A}, \tilde{B}, \tilde{C} \), Lemma 3.2 provides a convex approach to compute the optimal \( H_\infty \) filter. Substituting the expressions of the closed-loop system (8) into the matrix inequality (5), we arrive at the synthesis result in the variables \( A_f, B_f, C_f, D_f \) and possible auxiliary variables. However, the products \( P\tilde{A} \) and \( P\tilde{B} \) that appear in (5) are no longer linear in the decision variables. In this case, Lemma 3.2 cannot be directly invoked, and we now need to find a suitable congruence transformation and a change of variables that can convert (5) into an LMI. To this end, we partition the Lyapunov matrix \( P \) and \( P^{-1} \) as

\[
P = \begin{bmatrix} X & U \\ U' & \tilde{X} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} Y & V \\ V' & \tilde{Y} \end{bmatrix},
\]

with all the partitions having dimension \( n \times n \). By a block multiplication between \( P \) and its inverse, we obtain the following two important relations \( XY + UV' = I \) and \( UY + \tilde{X}V' = 0 \). Note that, for a positive definite matrix \( X \) and a non-singular matrix \( U \), it is always possible to find a matrix \( \tilde{X} > 0 \) such that \( P > 0 \). Moreover, the congruence transformation matrices defined as

\[
T = \begin{bmatrix} Y & V \\ X^{-1} & 0 \end{bmatrix}, \quad \tilde{T} = \begin{bmatrix} T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad TP = \begin{bmatrix} I & 0 \\ I & X^{-1}U \end{bmatrix}
\]

are non-singular. For \( \tilde{A} \) in (8) the product \( TP\tilde{A}T' \) is given by

\[
TP\tilde{A}T' = \begin{bmatrix} AY + B_uC_fV' & AX^{-1} \\ AY + B_uC_fV' + X^{-1}U_AfV' & AX^{-1} \end{bmatrix} = \begin{bmatrix} AY + B_uF & AZ \\ AY + B_uF + Q & AZ \end{bmatrix}.
\]
with new variables $Z = X^{-1}$, $Q = ZUA V'$, and $F = C_f V'$. Similarly, $TP\bar{B}$, $T\bar{C}$, and $TPT'$ are, respectively, given by

$$TP\bar{B} = \begin{bmatrix} B_w + B_u D_f \\ B_w + B_u D_f + G \end{bmatrix}, \quad T\bar{C} = \begin{bmatrix} YC' \\ ZC' \end{bmatrix}, \quad TPT' = \begin{bmatrix} Y & Z \\ Z & Z \end{bmatrix},$$

with the new variable $G = ZUB_f$.

The LMI s in the theorem now follow directly by applying the congruence transformation matrix $\bar{T}$ to both sides of (5), i.e., by multiplying (5) to the left by $\bar{T}$ and to the right by $\bar{T}^T$.

The reconstruction of $A_f$, $B_f$ and $C_f$ depends on the matrices $U$ and $V$. These are arbitrary matrices that must satisfy the relation $UV = I - Z^{-1}Y$. One possible choice for $U$ and $V$ is $U = Z^{-1} - Y^{-1}$ and $V = -Y$. For this particular choice, the filter parameters become

$$A_f = -(I - ZY^{-1})^{-1} Q Y^{-1}, \quad B_f = (I - ZY^{-1})^{-1} G, \quad C_f = -FY^{-1}. \quad \Box$$

**Theorem 3.5** (Convex $\mathcal{H}_2$ filtering problem). Let the system be given by (6) and let the filter be given by (7). Then, the following statements are equivalent:

(i) There exists a stable dynamic filter of order $n$ such that

$$\mu^* = \min_k \|T_{w}(s)\|_2^2.$$

(ii) The following LMI problem has a solution in the unknowns $W$, $Y$, $Z$, $G$, $Q$, $D_f'$:

$$\mu^* = \min \mu, \quad \text{Tr}[W] < \mu, \quad H_2 \leq 0, \quad \begin{bmatrix} Y & Z & YC' \\ Z & Z & ZC' \\ CY & CZ & W \end{bmatrix} > 0,$$

with the matrix $H_2$ given by (10). Once this problem is solved, the filter parameters $A_f$, $B_f$ and $C_f$ are, respectively, given by (12).

**Proof.** Similar to the proof of Theorem 3.4. $\Box$

3.3. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ state-feedback designs

Applying the control law given by $u(t) = K\hat{x}(t)$, the closed-loop system is given by $T_{w}(s) = \tilde{C}(sI - \tilde{A})\tilde{B}$ with $\tilde{A} = (A + B_u K)$, $\tilde{B} = B_w$, and $\tilde{C} = C$. Since we assume full state information, a dynamic compensator would not improve the performance.

**Theorem 3.6** (Convex $\mathcal{H}_\infty$ control problem). Let the plant model be described by (6). Define $Y = KP$. Then, the following statements are equivalent:

(i) There exists a stabilizing state-feedback gain $u(t) = K\hat{x}(t)$ such that

$$\eta^* = \min_k \|T_{w}(s)\|_\infty^2.$$

(ii) The following LMI problem has a solution $\eta^*$ in the unknowns $\eta, P, Y$:

$$\eta^* = \min \eta, \quad P > 0, \quad \begin{bmatrix} AP + PA^T + B_u Y + Y'B_u & B_w & PC' \\ B_w & -I & 0 \\ CP & 0 & -\eta I \end{bmatrix} < 0.$$

Once the above problem is solved, the control gain $K$ is given by $K = YP^{-1}$.

**Proof.** It follows from Lemma 3.2 and the change of variable $KP = Y$. $\Box$

**Theorem 3.7** (Convex $\mathcal{H}_2$ control problem). Let the plant model be described by (6). Define $Y = KP$. Then, the following statements are equivalent:

(i) There exists a stabilizing state-feedback gain $u(t) = K\hat{x}(t)$ such that

$$\mu^* = \min_k \|T_{w}(s)\|_2^2.$$
(ii) The following LMI problem has a solution $\mu^*$ in the unknowns $\mu, W, P, Y$:

$$
\mu^* = \min \mu, \quad \text{Tr}[W] < \mu, \quad \left[ \begin{array}{ccc}
P & PC^T \\
CP & W
\end{array} \right] > 0, \quad \left[ \begin{array}{ccc}
AP + PA' + B_uY + Y'B'_u & B_w \\
B'_w & -1
\end{array} \right] < 0.
$$

Once a solution is found, the control gain is given by $K = YP^{-1}$.

Proof. Follows directly from Lemma 3.1 and the change of variable $KP = Y$. \qed

3.4. Equivalence between controllers

We emphasize that the closed-loop performance of the proposed dynamic pre-filter and the static state-feedback controller are equivalent if the model has no uncertainty. The ideas behind this fact are found in [26], where, in a more general context, it is shown that the disturbance feedforward (DF) problem and the full information (FI) problem are equivalent in the sense that, given either the $K_{DF}$ controller or the $K_{FI}$ controller (obtained, respectively, from the DF and FI problems), one can construct the other in such a way that the closed-loop transfer functions are equivalent, i.e., $T_{DF} = T_{FI}$.

Our feedforward and feedback designs are special cases of the more general DF and FI problem. For the feedforward design, it is considered measurements signals (to be feed into the controller) in the form $y = w$, whereas the general DF problem assumes measurements in the form $y = Cx + w$. For the feedback case, it is considered a static state-feedback controller in the form $u = Ku$, whereas the general FI problem assumes measurements of the form $y = Cx$. For a two-degree-of-freedom dynamic controller given by $u = K(s)y$.

In our case, it can be proven that the feedback and feedforward designs are also equivalent. The proof in one direction can be obtained by noticing that the closed-loop transfer function from the exogenous input $w$ to the regulated (performance) output $z$ achieved by the static state-feedback gain $K$ is the same as the closed-loop transfer function from $w$ to $z$ obtained using the pre-filter $A_f = A + B_uK$, $B_f = B_w$, $C_f = K$, and $D_f = 0$, that is, $T_{wu}(s) = K(sI - (A + B_uK))^{-1}B_w$. The reverse direction can be proven by noticing that the optimal $H_2$ and a (central) suboptimal full information $H_\infty$ control law are actually pure static state-feedback.

3.5. Constraints on the control activity

In the previous derivation for the feedforward and feedback designs, there are no restrictions on the magnitude of the control effort. Thus, high gain controller typically results. However, it is possible to constrain the control activity. The idea is to bound the $H_2$ norm of the transfer function $T_{wu}(s)$ from the disturbance $w$ to the control input $u(t)$ by an amount $Y$. If the disturbance $w$ is an impulse, $Y$ is an upper bound on the total energy of the control input $u(t)$, since $\|T_{wu}(s)\|_2 = \|u(t)\|_2 = (\int_0^\infty u(t)^2 dt)^{1/2}$.

For the state-feedback case, where $u(t) = Kx(t)$, we define an additional regulated (performance) output $z = u(t)$. Thus, the transfer function from $w(t)$ to $u(t)$ is given by $T_{wu}(s) = C(sI - A)^{-1}B$ with $A = (A + B_uK)$, $B = B_w$, and $C = K$. Using Lemma 3.1, the bound $\|T_{wu}(s)\|_2^2 < Y$ holds if

$$
\text{Tr}[U] < Y; \quad \begin{bmatrix} Q & Y \\ Y & U \end{bmatrix} > 0, \quad \begin{bmatrix} \hat{A}Q_u + Q_u\hat{A}' & \hat{B} \\ \hat{B}' & -I \end{bmatrix} < 0,
$$

where we have used an auxiliary variable $U$ and the change of variable $Y = KQ_u$.

It is now possible to pose the $H_\infty$ state-feedback design with a bound $Y$ on the system gain $T_{wu}$. For this purpose, one needs to augment the original LMI conditions stated in Theorem 3.6 by adding the above LMI (14). However, for tractability in the LMI framework, we enforce a single Lyapunov matrix $P = Q_u$ for all sets of constraints (see [31]). In general, this introduces some conservatism. The LMI conditions become

$$
\eta^* = \min \eta, \quad P > 0, \quad \text{Tr}[U] < Y; \quad \begin{bmatrix} P & Y \\ Y & U \end{bmatrix} > 0, \quad \begin{bmatrix} AP + PA' + B_uY + Y'B'_u & B_w & PC^T \\ B'_w & -I & 0 \\ CP & 0 & -\eta I \end{bmatrix} < 0.
$$

Likewise, for the $H_2$ state-feedback design, the LMI conditions become

$$
\mu^* = \min \mu, \quad \text{Tr}[W] < \mu, \quad \text{Tr}[U] < Y; \quad \begin{bmatrix} P & PC^T \\ CP & W \end{bmatrix} > 0, \quad \begin{bmatrix} P & Y \\ Y & U \end{bmatrix} > 0, \quad \begin{bmatrix} AP + PA' + B_uY + Y'B'_u & B_w \\ B'_w & -I \end{bmatrix} < 0.
$$

For the $H_2$ and $H_\infty$ feedforward designs, we proceed in a similar manner. For tractability, we also enforce a single Lyapunov function. To have a strictly proper transfer function $T_{wu}(s)$, we let $D_f = 0$ in (7). The control law becomes $u(t) = -FY^{-1}x(t)$ since $C_f = -FY^{-1}$. Consequently, defining the regulated output $z(t) = u(t)$ is equivalent to defining the
output matrix \( \hat{C} \) as \( \hat{C} = [0 - FY^{-1}] \) in Eq. (8). From Lemma 3.1, the bound \( \|T_{ww}(s)\|_2^2 < \gamma \) holds if

\[
U^T \hat{C}^{-1} \hat{C}^T, \quad \text{Tr}[U] < \gamma.
\]

Using the definition for \( P^{-1} \) given in (11), it is easily verified that \( \hat{C}P^{-1} \hat{C}^T = FY^{-1} \hat{Y}Y^{-1}F^T \). Moreover, we further have that \( \hat{C}P^{-1} \hat{C}^T = F(Y-Z)^{-1}F^T \), since \( \hat{Y} = -U^{-1}XV = Y(Y-Z)^{-1}Y \). This last expression for \( \hat{Y} \) is promptly derived from the relations \( XV^T + U^TY = 0 \) and \( UV + XV = 0 \), and from the choices \( U = (X - Y^{-1}) > 0 \) and \( V = -Y < 0 \). Finally, using Schur complements, the bound on the control input \( u(t) \) can be expressed by

\[
\begin{bmatrix}
Y - Z & F^T \\
F & U
\end{bmatrix} > 0, \quad \text{Tr}[U] < \gamma.
\]

(15)

To solve the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) feedforward problems with constraint on the control activity, the LMI (15) must, respectively, be added to the LMI condition in Theorems 3.5 and 3.4.

3.6. The guaranteed cost control design

Using similar ideas as in [30], uncertainties described by bounded polyhedral domains can be incorporated into the proposed \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) feedback and feedforward designs. These robust compensators are called guaranteed cost controllers, since they minimize an upper bound for the worst-case performance. We now briefly describe the main ideas.

It is assumed that the system matrices \( M = (A, B_w, B_u, C) \) from (6) are unknown, but lie inside a given bounded polyhedral domain expressed as the convex combination of \( N \) given extreme matrices \( M_i = (A_i, B_{wi}, B_{ui}, C_i) \), for \( i = 1, \ldots, N \), that is, the unknown system \( (A, B_w, B_u, C) \) belongs to the polytope

\[
\mathcal{P} = \left\{(A, B_w, B_u, C) : (A, B_w, B_u, C) = \sum_{i=1}^{N} z_i (A_i, B_{wi}, B_{ui}, C_i) \right\},
\]

with \( z_i \geq 0 \) and \( \sum z_i = 1 \). It is clear that the knowledge of the value of \( z \) defines a precisely known system inside the polytope \( \mathcal{P} \), that is, \( M \in \mathcal{P} \) if and only if \( M = \sum z_i M_i \). The closed-loop transfer function from \( w \) to \( z \) using an uncertain model \( M \) from \( \mathcal{P} \) and a given compensator \( \mathcal{K} \) is denoted by

\[
T_M^X = (\mathcal{F}(G, \mathcal{K}) : G = C(sI - A)^{-1}B_w \text{ with } M \in \mathcal{P}),
\]

where \( \mathcal{F}_1 \) denotes the lower linear fractional transformation (LFT).

The robust \( \mathcal{H}_\infty \) and \( \mathcal{H}_2 \) filtering problem can now be stated as: find optimal filters \( \mathcal{F}_F \mathcal{H}_\infty \) and \( \mathcal{F}_F \mathcal{H}_2 \) that, respectively, minimize the guaranteed performance indexes \( \rho_\infty \) and \( \rho_2 \) such that

\[
\max_{M \in \mathcal{P}} \|T_M^X F_X\|_\infty^2 \leq \rho_\infty \quad \text{and} \quad \max_{M \in \mathcal{P}} \|T_M^X F_X\|_2^2 \leq \rho_2.
\]

In an analogous way, one defines the robust \( \mathcal{H}_\infty \) and \( \mathcal{H}_2 \) state-feedback control problem as: find optimal controllers \( \mathcal{K}_F \mathcal{H}_\infty \) and \( \mathcal{K}_F \mathcal{H}_2 \) that, respectively, minimize the guaranteed performance indexes \( \rho_\infty \) and \( \rho_2 \) such that

\[
\max_{M \in \mathcal{P}} \|T_M^X F_X\|_\infty^2 \leq \rho_\infty \quad \text{and} \quad \max_{M \in \mathcal{P}} \|T_M^X F_X\|_2^2 \leq \rho_2.
\]

It can be shown that a tractable (suboptimal) solution for the above problems are, respectively, obtained by solving the LMIs stated in Theorems 3.4–3.7, on the vertices of the polytope \( \mathcal{P} \). It should be emphasized that the resulting optimal controller might be conservative.

4. Numerical results

To be able to design the feedforward and the feedback compensators, proposed, respectively, in Sections 3.2 and 3.3, we shall identify a state-space model for the FRFs \( G_{zw}(j\omega) \) and \( G_w(j\omega) \) computed in Section 2. Since these FRFs are discrete numerical data, we first estimate an 8th order discrete-time model in the 80–310 Hz frequency range (as justified in Section 2) and subsequently we convert it into an equivalent 8th order continuous-time model. We do not present the system matrices since it would take considerable space.

4.1. State-space model estimation

In order to identify the discrete-time state-space model, we use the eigensystem realization algorithm (ERA) proposed in [33]. This algorithm estimates the parameters of the dynamic system using the impulse response data. Basically, it performs a singular value decomposition of the Hankel matrix constructed with these data. In our example, the impulse responses were obtained from the inverse Fourier transform of the numerical FRFs \( G_{zw}(j\omega) \) and \( G_w(j\omega) \). The estimated
The discrete-time model has the following realization:

\[ x_{k+1} = \bar{A}x_k + \bar{B}_w w_k + \bar{B}_u u_k, \]

\[ z_k = \bar{C}x_k. \]

Thus, the next step is to convert the discrete-time model (the above system matrices \( \bar{A}, \bar{B}_w, \bar{B}_u, \) and \( \bar{C} \) obtained from ERA) into equivalent continuous-time models \( G_w(s) = C(sI - \bar{A})^{-1}B_w \) and \( G_u(s) = C(sI - \bar{A})^{-1}B_u. \) The equivalent continuous-time model is obtained assuming a zero-order hold (ZOH) on the input.

We present in the left side of Fig. 3 (respectively, Fig. 4) the Bode magnitude plot of the continuous-time estimate \( G_w(s) \) obtained from the FRF \( G_w(j\omega) \) for the specific channel 1 (respectively, channel 16). Similarly, in the right side of Fig. 3 (respectively, Fig. 4), we show for channel 1 (respectively, channel 16) the data \( G_u(j\omega) \) and the Bode magnitude plot of its continuous-time model \( G_u(s). \)

For completeness sake, the Bode phase plots for the estimated models \( G_w(s) \) (solid line) and the FRFs \( G_w(j\omega) \) (dotted line) for channels 1 and 16 are shown in Fig. 5. We do not show the fitted FRFs for all the channels, since it would take considerable space. The fits are qualitatively similar for all the channels. It is always possible to improve the fitting using higher order models, but this significantly increases the computational burden without appreciable gain in performance. Note that the model is only supposed to fit the data in the frequency range between 80 and 310 Hz.
It is important to emphasize that the estimation and the conversion from discrete to continuous models approximate the true model with some error. In a practical implementation, it is usually desired to design robust controllers that can incorporate somehow these uncertainties. This can be achieved using the ideas described in Section 3.6.

The next Sections 4.2–4.4 apply the control techniques described in Section 3 to the previously obtained fitted models $G_w(s)$ and $G_u(s)$. In Section 4.2, we do not bound the control input $u$. Thus, this first experiment provides a measure of the best achievable limits of performance. In Section 4.3, on the other hand, we bound the control activity by an amount $U$.

We let the regulated (performance) output $z$ contain three equidistant microphones located in the far-field array (microphones 1, 16, and 31). For this purpose we use the notation $z = \{\text{mic}_1, \text{mic}_{16}, \text{mic}_{31}\}$. Once the controllers are obtained, we can analyze the closed-loop performance considering any set of outputs.

Throughout the next sections, the labeling is as follows: $\mathcal{H}_\infty$ stands for the dynamic $\mathcal{H}_\infty$ filter obtained as solution of the convex LMI problem (9) given in Theorem 3.4; $\mathcal{H}_2$ stands for the dynamic $\mathcal{H}_2$ filter obtained from the LMI problem (13) given in Theorem 3.5; the full state-feedback gains $\mathcal{K}_2$ and $\mathcal{K}_\infty$ are, respectively, obtained from Theorems 3.7 and 3.6; and $G_w$ stands for the estimated open-loop plant without control. The closed-loop system using these controllers are, respectively, denoted by $T_{\mathcal{H}_\infty}$, $T_{\mathcal{H}_2}$, $T_{\mathcal{K}_2}$, and $T_{\mathcal{K}_\infty}$. The notation for the guaranteed cost controller from Section 3.6 is the same but preceded by the letter $R$, e.g., $T_{R\mathcal{H}_2}$.

4.2. Control design without bounds on the control activity

In this first numerical experiment, we design the controller with no bounds imposed on the control input. The magnitude of the frequency response of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ feedforward compensators are presented in Fig. 6.
The optimal feedforward and feedback control signals are very large. This is expected since we did not bound the control activity.

We present in Fig. 7 the singular value plot of the frequency response of the open-loop plant $G_w$ and of the closed-loop systems $T_{wz}^{FH_1}$ (dashed line), $T_{wz}^{FH_2}$ (solid line), $T_{wz}^{KH_1}$ (×-mark), and $T_{wz}^{KH_2}$ (circle), considering channels 1, 16, and 31.

Table 1

<table>
<thead>
<tr>
<th>Controller</th>
<th>$|T_{wz}(s)|_2$</th>
<th>$|T_{wz}(s)|_\infty$</th>
<th>Open-loop $\mathcal{H}_2$ Performance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Performance</td>
<td>0.4535</td>
<td>-</td>
<td>0.8796</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>0.0175</td>
<td>0.1177</td>
</tr>
</tbody>
</table>

The optimal feedforward and feedback control signals are very large. This is expected since we did not bound the control activity.

We present in Fig. 7 the singular value plot of the frequency response of the open-loop plant $G_w$ and of the closed-loop systems $T_{wz}^{FH_1}$, $T_{wz}^{FH_2}$, $T_{wz}^{KH_1}$, and $T_{wz}^{KH_2}$ formed by connecting the respective compensators to the 8th order identified model. Recall from (1) that the largest singular value is just the $\mathcal{H}_\infty$ norm. This figure shows that the closed-loop system under the $\mathcal{H}_\infty$ full state-feedback controller possesses the same performance as the closed-loop system under the $\mathcal{H}_2$ feedforward compensator. This equivalence between the feedback and feedforward compensators was expected as explained in Section 3.4.

To generate this singular value plot, we have selected the output $z = (\text{mic}_1, \text{mic}_{16}, \text{mic}_{31})$, which is the same as the one used in the design of the controllers. For this specific output, Fig. 7 shows that the performance of the system improves considerably with active control, that is, the singular values decreased. We show later that the singular values provide a measure of the amount of sound power received by the microphones if we take the disturbance force $w$ to be an unit impulse. However, in the frequency range between 200 and 280 Hz, these controllers are not effective. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the open-loop plant $G_w$ and the closed-loop systems using the feedback and feedforward controllers are presented in Table 1. If we consider only microphones $z = (\text{mic}_1, \text{mic}_{16}, \text{mic}_{31})$, we see that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers provide a reduction from 0.8796 to 0.4535 in terms of $\mathcal{H}_2$ norm, and the $\mathcal{H}_\infty$ controllers provide a reduction from 0.1177 to 0.0175 in terms of $\mathcal{H}_\infty$ norm.

Fig. 8 presents the magnitude plot of the frequency response of the closed-loop system for the channels 1, 16 and 31. Observing these plots, one sees that active control significantly attenuates the sound pressure in those channels. This was expected, since the controllers were designed considering uniquely these channels as regulated (performance) output. One also sees that the sound pressure in channel 31 increased significantly in the frequency range between 200 and 280 Hz. This is in accordance with the singular value plot of Fig. 7.

Although these controllers were designed using only the information of microphones 1, 16 and 31 as the regulated output, it is also interesting to check what closed-loop performance they provide by looking at the singular value plots computed considering all 31 channels. This is illustrated in Fig. 9, where, one sees that the controllers still improve the performance. It is worth to emphasize that we cannot speculate about the behavior of these controllers outside the 80–310Hz frequency range.

The resulting values of $\mu$ and $\eta$ obtained in the control synthesis are identical to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms computed afterwards by analysis on the closed-loop systems.
written as \( W \) the formula thus captured by the microphones. This is the received sound power (RSP), in Watts per unit width. It can be computed by

\[
W = \frac{1}{2} \rho c \sum_{i=1}^{31} |z_i|^2 v_i
\]

where \( z_i \) is the pressure (Pa) at the microphone \( i \), the complex quantity \( v_i \) is the particle velocity (m/s) at the microphone \( i \), and \( A_i \) is the influential area for each microphone (the spacing between the microphones) given by \( A_i = 0.50/30 \text{m} \). Under free-field, plane wave assumptions, this expression can be equivalently written as

\[
W = \frac{A_s}{(2 \rho c)} \sum_{i=1}^{31} |z_i|^2 v_i
\]

where \( \rho \) is the air density and \( c \) is the speed of the sound. Now, if we take the disturbance force \( w \) to be an unit impulse, the output of our single input system is \( \mathcal{H}(\omega) = A_s/(2 \rho c) \sigma[G_w(j\omega)]^2 \). Therefore, the \( \mathcal{H}_\infty \) norm, the singular values, and the radiated sound power, yield qualitatively the same information. We can also define an attenuation of the received sound power (ARSP) by relating the RSP \( W_o \) for the open-loop plant with the RSP \( W_c \) for the closed-loop system with some controller "\( c \)" through the formula \( 10 \log 10(W_o/W_c) \). Naturally, the symbol \( c \) will denote either the \( \mathcal{H}_2 \) controller or the \( \mathcal{H}_\infty \) controller. Note that the notation for the feedback and feedforward compensators is abbreviated to just \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \), since they have identical performance. Considering all 31 channels, the ARSP under these controllers is presented in Fig. 10.

We also compute a cumulative sum of the attenuation of the received sound power (CARSP) by the formula \( 10 \log 10(\sum W_o/W_c) \), where \( \sum \) means cumulative sum. The \( \mathcal{H}_2 \) controllers provide a CARSP of 32.3 dB and the \( \mathcal{H}_\infty \) controllers provide a CARSP of 31.1 dB. A higher value means better performance. Thus, we can infer that the \( \mathcal{H}_2 \) controller performs slightly better than the \( \mathcal{H}_\infty \) controller. It is worth emphasizing that these results represent the limits of performance that can be achieved in this application by any feedback or feedforward compensator in the terms of the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms. Considering all the microphones, these norms are presented in Table 2. We see that the \( \mathcal{H}_{\mathcal{H}_2}/\mathcal{H}_2 \)

---

**Fig. 8.** Bode magnitude plot (in dBref = 20\( \mu \text{Pa}/\text{N} \)) of the open-loop plant \( G_o \) (dotted line) and the closed-loop systems \( T_{wz}^{\mathcal{H}_\infty} \) (dashed line), \( T_{wz}^{\mathcal{H}_2} \) (solid line), \( T_{wz}^{\mathcal{H}_2} \) (\( \times \)-mark), and \( T_{wz}^{\mathcal{H}_\infty} \) (circle), considering channels 1, 16 and 31.

**Fig. 9.** Singular value plot of the open-loop plant \( G_o \) (dotted line) and the closed-loop systems \( T_{wz}^{\mathcal{H}_\infty} \) (dashed line), \( T_{wz}^{\mathcal{H}_2} \) (solid line), \( T_{wz}^{\mathcal{H}_2} \) (\( \times \)-mark), and \( T_{wz}^{\mathcal{H}_\infty} \) (circle), considering all the channels.
controllers provide a reduction from 2.8311 to 1.3944 in terms of the $H_2$ norm, and the $KH_1 = FH_1$ controllers provide a reduction from 0.3812 to 0.0598 in terms of $H_1$ norm.

### 4.3. Control design with bounds on the control activity

In this section, we bound the system gain $T_{wu}$ as described in Section 3.5. The aim is to enforce $\|u(t)\|_2 = 1.8$ for a unit impulse applied at the disturbance channel $w$. After an exhaustive search, we found that the appropriated bounds on the control activity to enforce $\|u(t)\|_2 = 1.8$ are $\Upsilon = 1.8$ for the $KH_2$ and $FH_2$ designs and $\Upsilon = 3.95$ for the $KH_\infty$ and $FH_\infty$ designs. The magnitude plot of the frequency response of the $FH_\infty$ and $FH_2$ feedforward compensators are shown in
Comparing this figure with Fig. 6, one sees that the magnitude and the bandwidth of these controllers reduced significantly. The peak values are now $k_{FH1} = 0.0620$ and $k_{FH2} = 0.0779$.

The received sound power estimated by the singular value plot of the frequency response of the closed-loop systems $T_{wz}^{FH1}$ (dashed line), $T_{wz}^{FH2}$ (solid line), $T_{wz}^{KH1}$ (×-mark), and $T_{wz}^{KH2}$ (circle), considering all the channels, is shown in Fig. 12. With the control activity bounded, one sees that the $H_2$ (respectively, $H_1$) full state-feedback controller again possesses similar performance as the $FH_2$ (respectively, $FH_1$) feedforward compensator. Thus, even considering bounds on the control activity, the choice between feedforward and feedback is irrelevant in terms of performance if we have full information and a noise-free system.

The $H_2$ and $H_\infty$ norms of the open-loop plant and closed-loop system designed with bounds imposed on the control activity are shown in Table 3. If we consider only microphones $z = (\text{mic}_1, \text{mic}_{16}, \text{mic}_{31})$, one sees that the $H_2$ and $H_\infty$ full state-feedback controller again possesses similar performance as the $FH_2$ (respectively, $FH_1$) feedforward compensator. Thus, even considering bounds on the control activity, the choice between feedforward and feedback is irrelevant in terms of performance if we have full information and a noise-free system.

The $H_2$ and $H_\infty$ norms of the open-loop plant $G_w$ and the closed-loop systems $T_{wz}^{FH1}$, $T_{wz}^{FH2}$, $T_{wz}^{KH1}$, and $T_{wz}^{KH2}$ are shown in Table 3. If we consider only microphones $z = (\text{mic}_1, \text{mic}_{16}, \text{mic}_{31})$, one sees that the $H_2$ and $H_\infty$ full state-feedback controller again possesses similar performance as the $FH_2$ (respectively, $FH_1$) feedforward compensator. Thus, even considering bounds on the control activity, the choice between feedforward and feedback is irrelevant in terms of performance if we have full information and a noise-free system.

The $H_2$ and $H_\infty$ norms of the open-loop plant $G_w$ and the closed-loop systems $T_{wz}^{FH1}$, $T_{wz}^{FH2}$, $T_{wz}^{KH1}$, and $T_{wz}^{KH2}$ are shown in Table 3. If we consider only microphones $z = (\text{mic}_1, \text{mic}_{16}, \text{mic}_{31})$, one sees that the $H_2$ and $H_\infty$ full state-feedback controller again possesses similar performance as the $FH_2$ (respectively, $FH_1$) feedforward compensator. Thus, even considering bounds on the control activity, the choice between feedforward and feedback is irrelevant in terms of performance if we have full information and a noise-free system.

Fig. 11. Comparing this figure with Fig. 6, one sees that the magnitude and the bandwidth of these controllers reduced significantly. The peak values are now $\|F_{\infty}\|_\infty = 0.0620$ and $\|F_{H2}\|_\infty = 0.0779$.

The received sound power estimated by the singular value plot of the frequency response of the closed-loop systems $T_{wz}^{FH1}$, $T_{wz}^{FH2}$, $T_{wz}^{KH1}$, and $T_{wz}^{KH2}$, considering all the channels, is shown in Fig. 12. With the control activity bounded, one sees that the $H_2$ (respectively, $H_1$) full state-feedback controller again possesses similar performance as the $FH_2$ (respectively, $FH_1$) feedforward compensator. Thus, even considering bounds on the control activity, the choice between feedforward and feedback is irrelevant in terms of performance if we have full information and a noise-free system.

The $H_2$ and $H_\infty$ norms of the open-loop plant $G_w$ and the closed-loop systems $T_{wz}^{FH1}$, $T_{wz}^{FH2}$, $T_{wz}^{KH1}$, and $T_{wz}^{KH2}$ are shown in Table 3. If we consider only microphones $z = (\text{mic}_1, \text{mic}_{16}, \text{mic}_{31})$, one sees that the $H_2$ and $H_\infty$ full state-feedback controller again possesses similar performance as the $FH_2$ (respectively, $FH_1$) feedforward compensator. Thus, even considering bounds on the control activity, the choice between feedforward and feedback is irrelevant in terms of performance if we have full information and a noise-free system.

The $H_2$ and $H_\infty$ norms of the open-loop plant $G_w$ and the closed-loop systems $T_{wz}^{FH1}$, $T_{wz}^{FH2}$, $T_{wz}^{KH1}$, and $T_{wz}^{KH2}$ are shown in Table 3. If we consider only microphones $z = (\text{mic}_1, \text{mic}_{16}, \text{mic}_{31})$, one sees that the $H_2$ and $H_\infty$ full state-feedback controller again possesses similar performance as the $FH_2$ (respectively, $FH_1$) feedforward compensator. Thus, even considering bounds on the control activity, the choice between feedforward and feedback is irrelevant in terms of performance if we have full information and a noise-free system.

The $H_2$ and $H_\infty$ norms of the open-loop plant $G_w$ and the closed-loop systems $T_{wz}^{FH1}$, $T_{wz}^{FH2}$, $T_{wz}^{KH1}$, and $T_{wz}^{KH2}$ are shown in Table 3. If we consider only microphones $z = (\text{mic}_1, \text{mic}_{16}, \text{mic}_{31})$, one sees that the $H_2$ and $H_\infty$ full state-feedback controller again possesses similar performance as the $FH_2$ (respectively, $FH_1$) feedforward compensator. Thus, even considering bounds on the control activity, the choice between feedforward and feedback is irrelevant in terms of performance if we have full information and a noise-free system.

The $H_2$ and $H_\infty$ norms of the open-loop plant $G_w$ and the closed-loop systems $T_{wz}^{FH1}$, $T_{wz}^{FH2}$, $T_{wz}^{KH1}$, and $T_{wz}^{KH2}$ are shown in Table 3. If we consider only microphones $z = (\text{mic}_1, \text{mic}_{16}, \text{mic}_{31})$, one sees that the $H_2$ and $H_\infty$ full state-feedback controller again possesses similar performance as the $FH_2$ (respectively, $FH_1$) feedforward compensator. Thus, even considering bounds on the control activity, the choice between feedforward and feedback is irrelevant in terms of performance if we have full information and a noise-free system.

Table 3

<table>
<thead>
<tr>
<th>Controller</th>
<th>$H_2$</th>
<th>$H_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$FH_2$</td>
<td>0.4564</td>
<td>-</td>
</tr>
<tr>
<td>$FH_1$</td>
<td>-</td>
<td>0.0217</td>
</tr>
<tr>
<td>$KH_2$</td>
<td>1.4101</td>
<td>-</td>
</tr>
<tr>
<td>$KH_1$</td>
<td>-</td>
<td>0.0703</td>
</tr>
</tbody>
</table>

We also ran this experiment using tighter bounds on the control activity. In all cases, the feedforward and feedback designs provided identical performance.

4.4 Robust control design

In this section, we assume that our vibroacoustic model is not precisely known. For mechanical systems, it is common to consider that the mass of the system is precisely known and that all the uncertainties are parametric and mainly due to changes in the stiffness and damping of the system. Knowing the analytical model, it can be shown that there is a state-space representation affine in the stiffness and in the damping. Moreover, assuming that the uncertain parameters belong to a closed box, the system can be described by the polytopic model given in Section 3.6.
Fig. 13. Attenuation of the received sound power considering all the channels. Designs with bounds on the control activity. Positive values mean that the controller attenuated the received sound pressure.

In our case, it is not possible to explicitly describe the uncertainty set, since we do not have the analytical model of the system but rather a nominal estimated state-space representation of the dynamics. In this case, it is usually a very difficult task to obtain a good estimate for the uncertainties, certainly beyond the scope of this paper. Recent advances are found in [34,35]. A possible remedy, which provides a practical way to model the uncertain system, is obtained as follows. Let us assume that the nominal system is in the modal form, where the system matrix $A$ is block diagonal with the real eigenvalues $\sigma_i$ on the diagonal and the complex eigenvalues $\sigma_i + \sqrt{-1} \omega_i$ in 2-by-2 blocks of the form

$$A_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}$$

on the diagonal. Matrix $A$ can thus be partitioned as $A = A_\sigma + A_\omega$, with the partition $A_\sigma$ containing the $\sigma_i$ and the partition $A_\omega$ containing the $\omega_i$:

$$A = \begin{bmatrix} \sigma_1 \\ \sigma_i \\ \sigma_{i+1} \\ \vdots \\ \sigma_n \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_i \\ \sigma_{i+1} \\ \vdots \\ \sigma_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega_{n-1} \\ \omega_n \\ -\omega_{n-1} \end{bmatrix}.$$

In this form, the uncertain parameters are now the $\sigma_i$ and $\omega_i$, which are allowed to vary in the range $\sigma_i \pm \sigma$ and $\omega_i \pm \delta \omega$, from their nominal values $\sigma_i$ and $\omega_i$. A realistic assumption for the uncertainty range are $\sigma \approx 20\%$ and $\delta \omega \approx 5\%$ of their nominal values. The polytopic model from Section 3.6 can now be derived by noting that matrix $A$ lies inside the polytope described by the vertices $A_1, \ldots, A_4$ given by

$$A_1 = A_\sigma - \Sigma + A_\omega - \Omega, \quad A_2 = A_\sigma - \Sigma + A_\omega + \Omega,$$
$$A_3 = A_\sigma + \Sigma + A_\omega - \Omega, \quad A_4 = A_\sigma + \Sigma + A_\omega + \Omega,$$

with $\Sigma = 0.20A_\sigma$ and $\Omega = 0.05A_\omega$. Although the uncertainties also affect the other system matrices, we assume their effect are small and take the matrices $B_u$, $B_v$, and $C$ to be constant. Fig. 14 shows the singular value plot of this uncertain model, considering all the channels. This plot is generated using a fine grid on the parameter space.

The results using the guaranteed cost controller from Section 3.6 are now presented. Fig. 15 presents the uncertain closed-loop system using the $\mathcal{H}_\infty \mathcal{H}_2$ feedback controller (dark solid lines) and the $\mathcal{H}_\infty \mathcal{H}_2$ feedforward controller (light dotted line). Fig. 16 presents the closed-loop system using the $\mathcal{H}_\infty \mathcal{H}_\infty$ state-feedback controller (dark solid line) and the $\mathcal{H}_\infty \mathcal{H}_\infty$ dynamic feedforward controller (light dotted line). The systems in all these figures are computed through a fine grid on the parameter space.

One can easily see from Figs. 15 and 16 that the closed-loop systems using the feedback controllers have a more uniform (robust) behavior than the closed-loop systems using the feedforward controllers. This agrees with the fact that feedback compensators are less sensitive to uncertainties than the feedforward ones. This conclusion can also be observed from Table 4 that shows the differences among the robust designs. In this table, the notation $\max_\sigma \| \cdot \|_2$ (respectively, $\max_\sigma \| \cdot \|_\infty$) means the worst-case $\mathcal{H}_2$ norm (respectively, $\mathcal{H}_\infty$ norm) computed through a fine grid on the parameter space.
Fig. 14. Singular value plot of the (open-loop) uncertain model, considering all the channels. The nominal plant $G_w$ is shown in solid line.

Fig. 15. Singular value plot of the uncertain closed-loop systems using the $\mathcal{R}_2$ feedback controller (dark solid line) and the $\mathcal{R}_1$ feedforward controller (light dotted line), considering all the channels.

Fig. 16. Singular value plot of the closed-loop uncertain systems using the $\mathcal{R}_\infty$ feedback controller (dark solid line) and the $\mathcal{R}_\infty$ feedforward controller (light dotted line), considering all the channels.
space. The symbol $\rho$ indicates the guaranteed performance index for each of the designs. From this table, it is clear that the feedback controllers provide a better performance by providing smaller worst-case $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Note that this conclusion does not directly apply to the adaptive feedforward controllers commonly found in ANC applications. The emphasis in this section is on robust control design and not on adaptive control design.

An important fact shown in Table 4 is that all four performance bounds $\rho$ are not significantly larger than the computed worst-case $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. This leads to the conclusion that the designed robust controllers are not too conservative, although they have been computed using an approach based on quadratic stability criteria.

We have also computed the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed cost for the open-loop plant and the closed-loop uncertain models using the algorithms proposed in [36,37], which use parameter-dependent Lyapunov functions, providing less conservative estimates of the worst-case $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. They provide four algorithms to compute the $\mathcal{H}_2$ norm and three for the $\mathcal{H}_\infty$ norm. We have applied all the algorithms and have selected the smallest value for each norm. The results are presented in Table 5, where one sees that all values, except the $\mathcal{RFH}_2$ column, are the same as the worst-case values from Table 4.

### Table 4

<table>
<thead>
<tr>
<th>Performance</th>
<th>Open-loop</th>
<th>$\mathcal{RXH}_2$</th>
<th>$\mathcal{RFH}_2$</th>
<th>$\mathcal{RXH}_\infty$</th>
<th>$\mathcal{RFH}_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max r ||_2$</td>
<td>0.9820</td>
<td>0.4823</td>
<td>0.5365</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$\max r ||_\infty$</td>
<td>0.1465</td>
<td>–</td>
<td>–</td>
<td>0.0233</td>
<td>0.0355</td>
</tr>
<tr>
<td>$\rho$</td>
<td>–</td>
<td>0.4906</td>
<td>0.6035</td>
<td>0.0247</td>
<td>0.0377</td>
</tr>
</tbody>
</table>

### Table 5

<table>
<thead>
<tr>
<th>Guaranteed cost</th>
<th>Open-loop</th>
<th>$\mathcal{RXH}_2$</th>
<th>$\mathcal{RFH}_2$</th>
<th>$\mathcal{RXH}_\infty$</th>
<th>$\mathcal{RFH}_\infty$</th>
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<tr>
<td>$\mathcal{H}_2$</td>
<td>0.9838</td>
<td>0.4823</td>
<td>0.5492</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$\mathcal{H}_\infty$</td>
<td>0.1466</td>
<td>–</td>
<td>–</td>
<td>0.0233</td>
<td>0.0355</td>
</tr>
</tbody>
</table>

5. Conclusions

This paper presents $\mathcal{H}_2$ and $\mathcal{H}_\infty$ feedforward and feedback LMI-based control strategies for reducing noise due to structural vibration in a double panel system containing absorption material for acoustic insulation. The results for the different combinations of norms and control strategies are compared for an ideal system and a model with polytopic uncertainty. An $\mathcal{FH}_2$ (respectively, $\mathcal{FH}_\infty$) dynamic filter was designed to minimize the $\mathcal{H}_2$ (respectively, the $\mathcal{H}_\infty$) norm of the closed-loop system, with the constraint that it must be proper and stable. For this design, it is assumed that the disturbance acting on the system is measured. The $\mathcal{NH}_2$ and $\mathcal{NH}_\infty$ state-feedback controllers are also designed using LMIs. They assume full information of the state of the system. Considering no model uncertainties, the performance of the $\mathcal{NH}_2$ (respectively, $\mathcal{NH}_\infty$) state-feedback controller is equivalent to the $\mathcal{NH}_2$ (respectively, $\mathcal{NH}_\infty$) feedforward controller. Thus, in our setting, the choice of a feedforward or a feedback strategy would depend on what would be more convenient to measure, the disturbance or the state of the system. Once the class of compensator is chosen, the choice between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm must be intimately related with the assumptions on the disturbance acting on the system. Without much more effort, the LMI control designs are made robust to parametric uncertainties. In this case, the robust state-feedback designs are less sensitive to uncertainties than the dynamic feedforward designs.

We would like to stress that these designs are not implementable, since one cannot, in practice, perfectly measure neither the disturbance nor the complete state of the vibroacoustic system. However, the proposed control designs are valuable for several reasons. First, they can be used to assess the best possible closed-loop performance that can be achieved by pure non-adaptive feedback and feedforward controllers for the system. Second, the obtained state-feedback gain might be the first step towards the design of a dynamic output feedback controller in the form of a full order observer combined with a state-feedback law. With the proposed technique, it is also possible to investigate how the overall performance changes when the plant is not precisely known. A possible extension of this work would be to compare the proposed feedback and feedforward strategies with adaptive least mean squares (LMS) filters that are quite common in the active noise control. In a preliminary way, this issue was experimentally addressed in [38]. However, from the theoretical point of view, it is not even clear how to compare adaptive and non-adaptive controllers. One direction could be along the lines of [16]. This is certainly a very interesting and promising open question that the authors plan to investigate in the future using the results provided in this paper as a starting point.
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References


