“Convexifying” Linear Matrix Inequality Methods for Integrating Structure and Control Design

Juan F. Camino1; M. C. de Oliveira2; and R. E. Skelton3

Abstract: This paper presents a methodology in the linear matrix inequality (LMI) framework to jointly optimize the linear control law and the linear parameters in the structure. The method allows the mass matrix to contain free parameters, while employing LMI methods. The paper solves a structure design problem which bounds the covariance of selected outputs, such as interstory drifts and their velocities, in the presence of random excitations. In fact, the method simultaneously designs the structure and the controller, yielding a hybrid control. The proposed method also allows one to guarantee bounds on the peak response in the presence of bounded energy excitations. With minor modifications, the method can also guarantee bounds on the $H_\infty$ performance and many other convex performance criteria. The nonconvex problem is approximated by a convex one by adding a certain function to make the constraint convex. This “convexifying” function is updated with each iteration until the added convexifying function disappears at a saddle point of the nonconvex problem. This is a new contribution to both control theory and structure design.

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Introduction

The history of structure design can be characterized by four eras: In the first era, the design sought simply to oppose gravity—a statics problem. In the second era, the dynamic response was important. The third era sought to add control features to an existing structure design. In the fourth era, the design of the structure and the design of the controller are integrated so that the dynamics of the control system and the dynamics of the structure are cooperating, rather than competing, according to a selected performance objective. During the last two decades, the mathematical tools of control theory have produced algorithms which allow one to bound the dynamic response given a class of uncertain time varying disturbances. Such tools can now be used for structure design, even if no control is involved. In this context, our approach for structure design allows performance bounds on the dynamic response of the output, whereas the more standard structural design code focus on the static response and eigenvalues.

It is a well known fact that the design of the structure and the design of a controller for a given system are not independent. Consequently, it might happen that both the control design and the structural design are competing with each other in order to achieve some prescribed dynamic behavior. That is, more control energy than necessary might be required to achieve the objectives. This is an important consideration in the control of civil structures since the apparent necessity of large control forces have impeded the acceptability of control as a viable method to impose structure design. Simultaneous design (Skelton et al. 1992; Housner et al. 1997; Grigoriadis and Skelton 1998) can significantly improve the overall performance of the system, in the sense that either the new system (designed by this “hybrid” approach) would need a smaller amount of control energy to attain the same performance, or the new system would provide superior performance than the systems designed by standard techniques. Unfortunately, the simultaneous design of structure and controller is not in practice tractable, and results in a nonconvex optimization problem. The available algorithms are computationally expensive (see Onoda and Haftka 1987; Grandhi 1989; Jin and Sepulveda 1995; Yang and Chen 1996; and references therein), without guaranteeing a local minimum. It can be shown that the integrated structure and control design problem is equivalent to a decentralized output feedback control problem, which is well known to be hard to solve.

Following a two step redesign approach, one idea extensively used by Grigoriadis et al. (1996) and Skelton and Kim (1992) was as follows. In the first step, a controller for a given nominal structure was designed to meet some prescribed closed loop performance bound $\gamma$. In a second step, the structure and the controller were simultaneously redesigned in order to minimize the active control energy subject to the constraint that the closed loop system matrix is kept constant. This preserves the same level of performance $\gamma$ from one iteration to the next. The feature that makes the joint structure/control problem convex is the constraint that holds the closed loop system matrix constant. Based on this idea, the algorithm for solving the integrated control and structure problem can be stated as: (1) for a given nominal structure, design the controller; (2) redesign structure and controller (keeping the closed loop plant matrix constant); (3) with this new plant return
to Step 1. This constraint on the closed loop system reduces the redesign (Iteration 2) to a constrained convex quadratic programming problem. This was a significant improvement over the existing methodologies. A further improvement was given in Lu and Skelton (2000), where the writers considered more general structures and used the mixed $H_2/H_{\infty}$ performance criteria via a linear matrix inequality (LMI) framework, but in the redesign step they still needed the convexifying constraint of matching the system matrix.

A more direct approach via LMI to deal with the integrated structure and control design that does not impose constraints in the closed loop system matrices was used in Grigoriadis and Skelton (1998) and Grigoriadis and Wu (1997). The techniques proposed solve the two step redesign procedure by iterating between two convex subproblems posed as LMIs. The algorithm can be summarized as follows: While keeping the parameters of the structure fixed, solve a convex control problem; and in a second step, fix the Lyapunov matrix (which provides the controller) and optimize for the parameters of the structure. This approach has the drawback that it does not allow the mass matrix to be optimized and it may have a slow convergence, since the Lyapunov matrix in the structure redesign step is fixed.

There are few techniques available in the literature that allows one to treat the mass as an uncertain parameter (Skelton et al. 1992; Jin and Sepulveda 1995; Grigoriadis et al. 1996; Housner et al. 1997; among others), though these techniques are not in the LMI framework. Our algorithm has also the advantage of optimizing directly the physical parameters of the structure instead of optimizing an uncertain matrix $\Delta \mathbf{A}$ (Hsieh 1992; Grigoriadis and Skelton 1998), and at the end of the redesign, trying to find suitable physical parameters that match this uncertain matrix $\Delta \mathbf{A}$, which might not exist.

Our paper presents a new theory for the simultaneous design of structure and controller that improves the existing methodologies in two different ways. Our approach is completely posed in the LMI framework. Our algorithm has also the advantage of optimizing directly the physical parameters of the structure instead of optimizing an uncertain matrix $\Delta \mathbf{A}$ (Hsieh 1992; Grigoriadis and Skelton 1998), and at the end of the redesign, trying to find suitable physical parameters that match this uncertain matrix $\Delta \mathbf{A}$, which might not exist.

Here we define some notation. The superscript $(\cdot)^T$ and $(\cdot)^{-1}$ means, respectively, the transpose and the inverse of a matrix. The operator $\text{Tr}$ is the usual trace of a matrix. The operator $\text{diag}(\alpha_1, \ldots, \alpha_n)$ stands for a diagonal matrix whose entries are the elements $\alpha_1, \ldots, \alpha_n$. The function $\mathbf{e}[\cdot]$ is the expectation operator.

Problem Statement

A large class of dynamic systems in the field of mechanics and structures can be represented by a vector second-order differential equation of the form

$$M \ddot{q} + D \dot{q} + S q = f(t)$$

where $q \in \mathbb{R}^n = \text{vector of generalized coordinates}; M \in \mathbb{R}^{n \times n} = \text{mass matrix}; S \in \mathbb{R}^{n \times n} = \text{stiffness matrix};$ and $D \in \mathbb{R}^{n \times n} = \text{damping matrix (with only viscous damping force $D = D^T$, but if gyroscopic terms are presented, then $D$ is not symmetric)}$. The mass matrix is assumed to be symmetric and positive defined, $M = M^T > 0$. The vector $f(t)$ is an external force applied to the system. For our purpose, this force will represent the control input and the disturbance input actuating on the system. So $f(t)$ takes the form

$$f(t) = \hat{B}_u u(t) + \hat{B}_w w(t)$$

where $u(t) = \text{control signal to be determined and } w(t) = \text{exogenous disturbance}$. We cast our problem in the stochastic framework so that $w(t)$ is a white noise process. However, equivalent results apply when $w(t) \in \mathcal{L}_2$ [meaning that $w(t)$ is bounded in the sense of two norm. In other words $w(t)$ has a finite power spectrum].

Using a convenient change of variables given by $x = [q^T \ q^T]^T$, the second-order differential Eq. (1) is promptly written in the state space form

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & I \\ -S & -D \end{bmatrix} x + \begin{bmatrix} 0 \\ \hat{B}_u \end{bmatrix} u + \begin{bmatrix} 0 \\ \hat{B}_w \end{bmatrix} w$$

or equivalently

$$\dot{x} = A x + B_u u + B_w w$$

This is a descriptor representation for this system. This form is frequently adopted when the matrix $E$ is not invertible.

In our approach for the integrated control and structure design problem, the parameters of the structure which are available for optimization appear in the mass, the damping, and the stiffness matrices. A very important property assumed here is that the system Eq. (1) is affine in these parameters. By this affine representation we mean that

$$M(\eta) = M_0 + \sum_j \eta_j M_j, \quad D(\beta) = D_0 + \sum_j \beta_j D_j, \quad \text{and}$$

$$S(\gamma) = S_0 + \sum_k \gamma_k S_k$$

where matrices $M_j, D_j,$ and $S_k$ is given. Since matrix $A$ in Eq. (2) is affine in $D$ and in $S$, and matrix $E$ is affine in $M$, the system (2) can also be written as

$$\dot{x} = A(\alpha)x + B_u u + B_w(\alpha) w$$

for $A(\alpha)$ and $E(\alpha)$ affine matrices given by

$$A(\alpha) = A_0 + \sum_i \alpha_i A_i, \quad E(\alpha) = E_0 + \sum_i \alpha_i E_i$$

$$B_u(\alpha) = B_{u0} + \sum_i \alpha_i B_{ui}$$

where the variable $\alpha$ contains, in a convenient way, the variables $\eta, \beta,$ and $\gamma$, i.e., $\alpha = (\eta, \beta, \gamma)$. Notice that it is paramount to adopt the descriptor representation (2) in order to preserve the affine property of the mass matrix $M$, consequently, allowing the mass of the system to be incorporated into the optimization problem.

Remark. To apply our methodology we do not need explicitly to assume the second-order representation. Any descriptor system which is affine in the parameters can be used. We use the second-order representation since we are mainly concerned with mechanical systems and structures.
**Integrated Structural Control**

For simplicity of presentation, we assume full state feedback law, with the control gain given by $u(t) = Kx(t)$. The derivation for the output feedback case does not require much more sophistication. The output vector for performance evaluation is

$$z(t) = C_x x(t)$$ (3)

We first present Theorem 1 which characterizes the control problem for the structure and control design, with a stochastic interpretation; later, we show its equivalence to the standard $H_2$ problem. The exogenous disturbance $w(t)$ applied to the system is assumed to be a stochastic white noise process with intensity $W = W^T > 0$, i.e., $e[w(t)w(t)^T] = W$. The performance criteria is to minimize the variance of the control $u(t)$ applied to the system, while the output $z(t)$ is bounded in the sense

$$\lim_{t \to \infty} e[z(t)z(t)^T] < \Omega$$ (4)

for some given positive definite matrix $\Omega$.

**Theorem 1.** Assume that the disturbance $w(t)$ is a stochastic white noise process with intensity $W = W^T > 0$. Define $F = KP$. Let $\Omega$ be a given positive definite matrix, and consider the descriptor system given in Eq. (2). Then the following statements are equivalent:

1. There exists structure parameter $\alpha$, and a stabilizing state feedback gain $u(t) = Kx(t)$ such that
   $$\lim_{t \to \infty} e[z(t)z(t)^T] < \Omega$$
   and
   $$\lim_{t \to \infty} e[u(t)^Tu(t)] < \gamma$$

2. There exists matrices of compatible dimensions $P = P^T > 0$, $U = U^T > 0$, and $F$, and parameter $\alpha$, such that the following inequalities are satisfied:
   $$\begin{bmatrix}
   A(\alpha)PE(\alpha) + E(\alpha)PA(\alpha)^T + B_u FE(\alpha) \\
   + E(\alpha)F^TB_w^T \\
   B_w(\alpha)^T \\
   - W^{-1}
   \end{bmatrix} < 0$$ (5)
   $$\text{Tr}(U) < \gamma, \begin{bmatrix}
   U & F \\
   F^T & P
   \end{bmatrix} > 0, \begin{bmatrix}
   \alpha & C_z & P
   \end{bmatrix} < 0$$ (6)

3. For some constant matrix $G$, there exists matrices of compatible dimensions $Q = Q^T > 0$, $U = U^T > 0$, and $K$, and parameter $\alpha$, such that the following LMI are satisfied:
   $$\begin{bmatrix}
   B_w(\alpha)^T \\
   - W^{-1} \\
   0 \\
   -Q
   \end{bmatrix} < 0$$ (7)
   $$\begin{bmatrix}
   A(\alpha)^T + KF^T \\
   0 \\
   0 \\
   -Q
   \end{bmatrix} < 0$$ (8)

$$\text{Tr}(U) < \gamma, \begin{bmatrix}
   U & K \\
   K^T & Q
   \end{bmatrix} > 0, \begin{bmatrix}
   \Omega & C_z \\
   C_z^T & Q
   \end{bmatrix} > 0$$ (9)

where $\mathbf{(*)}$ refers to the term

$$- [A(\alpha) + B_u K - E(\alpha)] G^T - G[A(\alpha) + B_u K - E(\alpha)]^T + G Q G^T$$

We present the proof of this theorem after the necessary tools provided by the convexifying algorithm have been introduced.

If the matrices $A$ and $E$ do not depend on the structure parameter $\alpha$, then the constraint (5) in Item (2) is an LMI hence a convex set, in $U$, $P$, and $F$. In other words, if the structure is known, the problem reduces to a standard convex state feedback control problem. If the matrices $A$ and $E$ depend on $\alpha$, then the product $A(\alpha)PE(\alpha) + B_u FE(\alpha)$ is nonlinear in the decision variables $\alpha$, $F$, and $P$. In this case, it is hard to find a solution. Even for the pure structural passive design case, where the control gain $K$ is given, the problem is still nonconvex.

Note that independently of the control parameter $K$, the product of the system matrix $A(\alpha)$ and the “Lyapunov” matrix $P$ are always present in Eq. (5). When the mass matrix $M$ is fixed (matrix $E$ does not depend on the parameter $\alpha$), the procedure adopted in Grigoriadis and Skelton (1998) and Grigoriadis and Wu (1997) is to iterate between two convex subproblems: first, for fixed structure parameter $\alpha$ solve for the Lyapunov matrix $P$ in Eq. (5) and second, for fixed matrix $P$ solve for the parameter of the structure $\alpha$ in Eq. (5). This strategy in practice converges slowly to a solution, although there is no guarantee for a local minimum.

The algorithm we propose in this paper also iterates between two subproblems, but in a more elaborate way. Before iterating, we apply some convexifying potential functions (see de Oliveira et al. 2002) to the nonconvex constraint in order to generate the conditions in Item (3). Notice that, for a constant matrix $G$, these conditions are simultaneously affine in the variables $Q$, $U$, $K$, and $\alpha$. In this sense, the joint structure/control problem has been “convexified.” Therefore, there will be no need to fix the Lyapunov matrix $P$ in the redesign step (instead, the fixed matrix will be the added potential matrix $G$). The convexifying potential method and the algorithm will be detailed in the next section.

**Remark.** The relation between the control problem presented in Theorem 1 and the $H_2$ control problem (see Boyd et al. 1994; Skelton et al. 1998) is stated in the next lemma.

**Lemma 2.** [$H_2$ Control Problem] Assume that the disturbance $w$ belongs to space $L_2$. Then the $H_2$ norm of the closed loop transfer function $H_{w,c}(s) := C_s [sI - E(\alpha)^{-1}A_c(\alpha)] E(\alpha)^{-1}B_w(\alpha)$, where $A_c(\alpha) = A(\alpha) + B_u K$, is bounded by $\sqrt{\text{Tr} \Omega}$, i.e., $\|H_{w,c}(s)\|_2 < \sqrt{\text{Tr} \Omega}$ if and only if the constraints (5)–(6) and (7)–(8) in Theorem 1 are feasible for $\gamma \to \infty$.

**Convexifying Algorithm**

In this section, we shall present the theory behind the convexifying algorithm, which is a powerful tool for solving control problems with structure imposed on the controller. For a detailed presentation of the convexifying potential functions applied to control theory see de Oliveira et al. (2000). In the integrated design problem stated herein, we do not impose constraints on the control gain matrix, although the control law could be subject to arbitrary affine structural constraints, enabling one to solve complex joint structure/control design problems. However, it is possible to show that the free structural parameters create the equivalence of a decentralized control problem where the “control” gain matrix is diagonal. In order to elaborate more on this point define

$$\Delta = \begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_i I \\
\vdots \\
\alpha_m I
\end{bmatrix}, \quad I = [I \cdots I \cdots]$$

$$A = \begin{bmatrix}
A_1 \\
\vdots \\
A_i \\
\vdots \\
A_m
\end{bmatrix}, \quad B_u = \begin{bmatrix} B_{u1} \\
\vdots \\
B_{ui} \\
\vdots \\
B_{um}
\end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\
\vdots \\
E_i \\
\vdots \\
E_m
\end{bmatrix}$$
Then the matrices $A(\alpha)$, $B_w(\alpha)$, and $E(\alpha)$ can be written as

$$A(\alpha) = A_0 + 1\Delta A$$
$$B_w(\alpha) = B_{w0} + 1\Delta B_w$$
$$E(\alpha) = E_0 + 1\Delta E$$

and the system Eq. (2) as

$$(E_0 + 1\Delta E)x = (A_0 + 1\Delta A)x + B_wKx + (B_{w0} + 1\Delta B_w)w$$

which after some manipulation gives

$$E_0\dot{x} = (A_0 + B_wK)x + l\ddot{u} + B_{w0}w$$

with

$$u_1 = \Delta y_1\quad y_1 = Ex$$
$$u_2 = \Delta y_2\quad y_2 = Ax$$
$$u_3 = \Delta y_3\quad y_3 = B_ww$$

and $\ddot{u}$ given by $\ddot{u} = u_2 + u_3 - u_1$.

If $y_i$ were a set of available measurements, and $\Delta$ were a diagonal control gain multiplying the measurements $y_i$, then the structure design problem has the same mathematical structure as a problem where the “control” signal $u_i$ depends only on the $i$th “measurement” signal $y_i$. In control jargon, this is called “decentralized” control. In general, the imposition of structure in the control gain matrix is called a “decentralized control problem,” and the mathematics needed to solve this problem are the same (find diagonal $\Delta$) as the structural design problem posed herein.

In the sequel, we propose an algorithm to find stationary solutions to the nonconvex structure/control problem using the idea of the convexifying potential functions developed in de Oliveira et al. (2000). In that paper, it was shown that many standard control problems such as $H_2$ and $H_{\infty}$ problems with some imposed structure in the controller can be formulated as an LMI problem having an extra nonconvex constraint. To use convex tools, convexifying potential functions were added to the nonconvex constraint. A convexifying algorithm to solve this nonconvex problem was also proposed. This algorithm iterates in a sequence of convex subproblems which, given some conditions, is guaranteed to converge to a stationary point of the original nonconvex problem.

Let $x$, $\eta$ belong to a convex set $\phi$, and $F(x)$ be a nonconvex matrix function. A convexifying potential function is a differentiable function $G(x, \eta)$ such that the function $F(x) + G(x, \eta)$ for all $x$, $\eta \in \phi$ is convex in $x$. Thus, if $F(x)$ satisfies certain conditions, a stationary point of the nonconvex optimization problem

$$\ddot{x} = \arg\min_{x \in \Omega} f(x), \quad \Omega = \{x \in \phi | F(x) < 0\}$$

(9)

can be obtained by iterating on a sequence of convex subproblems given by

$$x_{k+1} = \arg\min_{x \in B_k} f(x), \quad B_k = \{x \in \phi | F(x) + G(x, x_k) < 0\}$$

(10)

To impose that the optimality conditions of both optimization problems (9) and (10) are identical, the potential function $G$ must possess some extra properties such as being non-negative definite and that $G(x, \eta) = 0$ if, and only if, $x = \eta$.

In order to apply the above idea to the integrated structure and control design, we shall define the nonconvex function $F$ we are interested in. From the set of conditions (2) given in Theorem 1 we have that the constraint with nonlinear terms is Eq. (5). Completing the squares, this inequality can be manipulated into

$$\begin{bmatrix}
A_{cl}PE + EPA_{cl}^T & B_w \\
B_w^T & -W^{-1}
\end{bmatrix} < 0$$

$$= \begin{bmatrix}
A_{cl}PEA_{cl}^T + EPE^T - (A_{cl} - E)(A_{cl} - E)^T & B_w \\
B_w^T & -W^{-1}
\end{bmatrix} < 0$$

where $A_{cl}(\alpha, K) = A(\alpha) + B_wK$. The dependence of the matrices $A_{cl}$, $B_w$, and $E$ on $\alpha$ and $K$ is suppressed to simplify the notation.

Let $x = (P, \alpha, K)$ and define $Q = P^{-1}$, then by applying Schur complements we can rewrite the above inequality into the equivalent form

$$F(x) := \begin{bmatrix}
-(A_{cl} - E)(A_{cl} - E)^T & B_w & A_{cl} & E \\
B_w^T & -W^{-1} & 0 & 0 \\
A_{cl}^T & 0 & -Q & 0 \\
E^T & 0 & 0 & -Q
\end{bmatrix} < 0$$

(11)

Using this concave inequality to define $F$ it will be possible to guarantee convergence of the convexifying algorithm to a stationary point of the original nonconvex problem.

Now we shall define a potential convexifying function $G$ that makes $F + G$ convex. For this purpose, let $G$ be given by

$$G(x) := (A_{cl}P - E)P$$

and define the convexifying potential function $G(x, \eta)$ for the above nonconvex matrix function $F$ to be the positive semidefinite expression given by

$$0 \leq G(x, \eta) := (A_{cl} - E - G(\eta)P^{-1}) \left[(A_{cl} - E - G(\eta)P^{-1})^Tight] < 0$$

(12)

This function satisfies the convexifying assumptions, since it is positive semidefinite and Eq. (12) attains zero for the choice of matrices $\eta = \eta$, that is, $G(x, \eta) = 0$ for $x = \eta$. Moreover, adding Eq. (12) to the first block of Eq. (11), we obtain the following LMI:

$$\begin{bmatrix}
-(A_{cl} - E)G + G(A_{cl} - E)^T + GG^T & B_w & A_{cl} & E \\
B_w^T & -W^{-1} & 0 & 0 \\
A_{cl}^T & 0 & -Q & 0 \\
E^T & 0 & 0 & -Q
\end{bmatrix} < 0$$

The dependence of $G$ on $\eta$ has been also suppressed for simplicity. Substituting $A_{cl} = A(\alpha) + B_wK$ into the above inequality one recovers inequality (7) given in Theorem 1. In this form, the Lyapunov matrix $P = Q^{-1}$ and the system matrices $A(\alpha)$ and $E(\alpha)$ no longer appear as products. Instead, these products have been replaced by products with $G(\eta)$, which has been introduced with the potential function. Notice that $\eta$ is kept constant and equal to $\eta = \eta_k$ in the convex subproblems to be solved of the form (10).

Considering as the objective function to be minimized an upper bound on the covariance of the control energy, that is, $f = \gamma > \epsilon [u(t)^T u(t)]$, the ideas explained so far are summarized in the algorithm below.
Convexifying Algorithm for Structural Control

Let $f = \min \gamma$.

Set the nominal values for $\alpha_0, A_0$, and $E_0$.

Compute $K_0$ and $P_0$ by finding a feasible solution to
the convex conditions given in Item 2 of Theorem 1.

Set $\epsilon$ to some prescribed tolerance and $k = 0$.

**Repeat**

Set $G_{k} \leftarrow \left.A_{k}^T(w) - E(\alpha_k)\right|_P$.

For fixed $G = G_{k}$, solve $f_k = \min \gamma$ subject to Eqs. (7) and
(8) for $\alpha$, $K$, $Q$, and $U$.

Denote the solution $(\alpha^*, K^*, Q^*, U^*)$.

Set $(P_{k+1}, \alpha_{k+1}, K_{k+1}) \leftarrow (Q^{-1}, \alpha^*, K^*)$.

$k = k + 1$.

**Until** $\|f_k - f_{k-1}\| < \epsilon$.

It is possible to add an extra step to the above algorithm:
Before setting $G_{k}\leftarrow\left.A_{k}^T(w) - E(\alpha_k)\right|_P$, we update $P_k$ by solving
the LMI (7)–(8) with $\alpha = \alpha_k$. In our experiments, this extra
step provides a faster convergence of the convexifying algorithm
for structural control (CASC), although we do not guarantee a
stationary solution.

**Proof of Theorem 1**

This section provides the technical details needed for the proof of
Theorem 1.

**Proof.** The discussion in the previous section can be used to
to show the equivalence between Conditions (2) and (3) given in
Theorem 1. If the constraints in Item (2) have a feasible solution
$x \equiv (\bar{P}, \bar{a}, \bar{F}, \bar{P}^{-1})$, then $\mathcal{F}(x) < 0$. Hence the constraints in Item (3)
also have a feasible solution since $\mathcal{G}(x, x) = 0$ by definition of the
convexifying function. Conversely, if the constraints Item (3)
have a feasible solution $\bar{x}$, then $\mathcal{F}(\bar{x}) + \mathcal{G}(\bar{x}, \bar{x}) < 0$. But since
by definition $\mathcal{G}(\bar{x}, \bar{x}) = 0$, we have that $\mathcal{F}(\bar{x}) = \mathcal{F}(\bar{x}) + \mathcal{G}(\bar{x}, \bar{x})
< 0$, which implies that the constraints in Item (2) are also feasible.

The equivalence between Conditions (1) and (2) in Theorem 1
is provided by the following argument. Since, by assumption,
the mass matrix $M(\alpha)$ is positive definite for all $\alpha$ of interest,
the matrix $E(\alpha)$ is invertible. Hence, for the state feedback law given
by $u(t) = Kx$, the closed loop descriptor system

$$E(\alpha) \dot{x} = [A(\alpha) + B_u K]x + B_w(\alpha)w$$

$$E(\alpha) \dot{x} = \Lambda_c(\alpha) x + B_w(\alpha) w$$

can be equivalently written in standard state space form as

$$\dot{x} = E(\alpha)^{-1} \left[A_c(\alpha)x + B_w(\alpha)w\right]$$

$$\dot{x} = \Lambda_c(\alpha)x + B_w(\alpha)w$$

For the above stable state space system it is a standard result
(Skelton et al. 1998) that the conditions in Item (1) of Theorem 1
hold if, and only if

$$e[u(t)^T u(t)] = \text{Tr}(KP) < \gamma$$

and

$$e[z(t)^T z(t)] = C_z PC_z^T < \Omega$$

where the symmetric and positive definite matrix $P$ is a feasible
solution to the Lyapunov inequality

$$\Lambda_c(\alpha) P + P \Lambda_c(\alpha)^T + B_w(\alpha)W B_w(\alpha)^T < 0$$

This matrix $P$ is an upper bound to the closed loop controllability
Grammian. Applying a congruence transformation (which pre-
serves the inertia of the inequality) by multiplying on the left and
on the right side by the symmetric matrix $E(\alpha)$, we obtain the
equivalent inequality

$$A_c(\alpha) PE(\alpha) + E(\alpha) PA_c(\alpha)^T + B_w(\alpha)WB_w(\alpha)^T < 0$$

Using a Schur complement, it is possible to show that the above
inequality is equivalent to

$$\begin{bmatrix}
A_c(\alpha) PE(\alpha) + E(\alpha) PA_c(\alpha)^T & B_w(\alpha) \\
B_w(\alpha)^T & -W^{-1}
\end{bmatrix} < 0$$

(13)

Noting that $A_c(\alpha) = A(\alpha) + B_u K$ and $F = KP$, inequality (13)
becomes inequality (5) given in Item (2). To show Eq. (6), we
introduce the auxiliary symmetric variable $U$ such that

$$U > KP K^T = FP^{-1} F^T$$

then $\gamma > \text{Tr}(U) > \text{Tr}(KP K^T)$. Hence, using Schur’s complement,
this inequality is equivalent to

$$\text{Tr}(U) < \gamma, \begin{bmatrix} U & F \\ F^T & P \end{bmatrix} > 0$$

This completes the proof.

**Examples**

To illustrate the proposed methodology for solving the integrated
control and structure optimization problem in the LMI framework,
we walk through a simple example. For this purpose, we choose
the problem of isolating a civil engineering structure
against earthquakes. This will not be a comprehensive presenta-
tion on how to solve this specific structure problem, but rather on
providing efficient tools for this purpose.

**Table 1. Nominal Structural Parameters**

<table>
<thead>
<tr>
<th>Floor masses</th>
<th>Stiffness coefficients</th>
<th>Damping coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>[kN/m]</td>
<td>[kN/m]</td>
<td>[kN s/m]</td>
</tr>
<tr>
<td>$m_1 = 5,897$</td>
<td>$k_1 = 33,732$</td>
<td>$d_1 = 67$</td>
</tr>
<tr>
<td>$m_2 = 5,897$</td>
<td>$k_2 = 29,093$</td>
<td>$d_2 = 58$</td>
</tr>
<tr>
<td>$m_3 = 5,897$</td>
<td>$k_3 = 28,621$</td>
<td>$d_3 = 57$</td>
</tr>
</tbody>
</table>

Fig. 1. 3-DOF model
The field of controlling vibrations of structures against earthquakes has attracted the interest of many researchers. The references Kose et al. (1998); Ramalho et al. (2000); Spencer Jr. et al. (1998) provide a concise explanation of the structural problem, and a benchmark comparison of various structural control algorithms applied in an evaluation model obtained from experimental data.

The model of the system in consideration is shown in Fig. 1. This is a three-degrees-of-freedom version of the same structure used in Ramalho et al. (2000). The nominal values for this system are given in Table 1. The control inputs $u_i$ are independent forces applied to each floor. Hence, for the model (2), $\hat{B} = I$, the matrix $\hat{B} w$ is given by $\hat{B} w = (m_1, m_2, m_3)^T$, and the disturbance vector $w$ is assumed to be a white noise process with intensity $W = 16$ [m$^2$/s$^4$], which represents the earthquake acceleration of the ground motion $\hat{x}_g$ in Fig. 1.

For this dynamic system (2), the mass matrix $M$ is given by $M = \text{diag}(m_1, m_2, m_3)$, and the stiffness matrix $S$, and the damping matrix $D$ are given by

$$
S = \begin{bmatrix}
  k_1 + k_2 & -k_2 & 0 \\
  -k_2 & k_2 + k_3 & -k_3 \\
  0 & -k_3 & k_3
\end{bmatrix}
$$

$$
D = \begin{bmatrix}
  d_1 + d_2 & -d_2 & 0 \\
  -d_2 & d_2 + d_3 & -d_3 \\
  0 & -d_3 & d_3
\end{bmatrix}
$$

We seek to bound the output variances such that $\varepsilon z_i^2 \approx 0.0002$ [m] and $\varepsilon z_2^2 \approx 0.3$ [m/s], for $i = 1, 2, 3$. Thus, the diagonal of the output covariance matrix $C_z P C_z^T$ of the closed loop system should be bounded by $\Omega$, that is $C_z P C_z^T \leq \Omega$, with the bound $\Omega$ given by

$$
\Omega = (2 \times 10^{-4}, 2 \times 10^{-4}, 2 \times 10^{-4}, 0.3, 0.3, 0.3)
$$

In our notation, $C_z^T$ means the $i$th row of the matrix $C_z$. Note that $C_z P C_z^T \leq \Omega_i$ is a convex constraint.

The states are the displacement and the velocity of each floor relative to the ground, i.e., $q_i$ represents the displacement of the mass $m_i$, and $\dot{q}_i$ its velocity.

We are interested in the simultaneous design of the parameters of the structure and the controller, using the control implementation stated in Theorem 1. We seek designs which limit the variance of the interstory drift $z_i = q_i, z_{i+1} = q_{i+1} - q_i, i = 1, 2,$ and their velocities $\dot{z}_{i+3} = \dot{z}_i$. Thus, the output $C_z$ is given by

$$
C_z = \begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  -1 & 1 & 0 & 0 & 0 & 0 \\
  0 & -1 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & -1 & 1 & 0 \\
  0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}
$$
For all runs the stopping criteria was the relative error on the control energy between two successive iterations (\(\varepsilon[u^T u_{k+1}] - \varepsilon[u^T u_k]/\varepsilon[u^T u_k]\) is less than \(5 \times 10^{-4}\) for ten consecutive times.

We assume that the lower and the upper bounds on all the parameters are 0.5 and 2.0 of the nominal values in Table 1. For brevity, we will call the standard deviation of the control \(\varepsilon[u^T u]\) the "control effort."

**Example 1.** \((k_2, d_2)\). In this first example, the parameters to be redesigned are the spring stiffness \(k_2\) and the damping coefficient \(d_2\). We found by an exhaustive search the global optimum for the integrated structure and control design problem. These optimal values are \(k_2 = 26,699\) [kN/m] and \(d_2 = 116\) [kN s/m], which give the global minimal control effort \(\sqrt{\varepsilon[u^T u]} = 3,276,393\) [kN] required to achieve the design output performance \(\Omega\).

Now, we simulate our CASC algorithm. First, an initial controller \(K_0\) using the nominal parameters in Table 1 is determined, by solving the LMIs (5)--(6) in Theorem 1 (which for fixed \(\alpha\) it is a convex problem). The initial controller evaluated in this way is

\[
K_0 = \begin{bmatrix} -176,699 & -319,360 & -8,245 & -11,994 & -9,657 & -5,625 \\ 203,413 & -134,380 & -189,180 & -9,614 & -16,822 & -16,588 \\ 33,250 & 47,538 & -186,040 & -5,598 & -16,624 & -27,045 \end{bmatrix}
\] (14)

For this initial controller, the control effort is \(\sqrt{\varepsilon[u^T u]} = 4,408,517\) [kN]. Note that if the designer is not allowed to change the parameters of the structure, then the initial controller \(K_0\) provides the best required performance using the least control effort, which is much higher than the globally optimal effort given by \(\sqrt{\varepsilon[u^T u]} = 3,276,393\) [kN].

We proceed with the algorithm CASC (integrated design) generating the results in Fig. 2. After 26 iterations the algorithm converged to the solution \(k_2^* = 26,751\) [kN/m], \(d_2^* = 116\) [kN s/m], and the control gain...
which provides a control effort of $\sqrt{\text{trace}(u^T u)} = \sqrt{3,276,453}$ [kN].

For this controller $K^*$, the achieved output performance (Table 3) is

$$C_1^*P C_1^T = \begin{bmatrix}
-160,292.2 & -148,749.9 & -702.4 & -9,389.4 & -7,226.0 & -5,614.1 \\
187,423.4 & -175,483.0 & -149,112.4 & -7,176.1 & -14,506.8 & -15,039.6 \\
93,662.9 & -77,905.2 & -134,451.6 & -5,640.8 & -15,032.6 & -22,546.2
\end{bmatrix}$$

This shows that the constraints $\varepsilon z_1^2$, $\varepsilon z_2^2$, and $\varepsilon z_3^2$ are binding (active).

-----

**Example 2.** ($k_2, d_2, m_2$). We start this example with the same initial controller $K_0$, but we add the additional parameter $m_2$ to be optimized. The results from the integrated design is presented in Fig. 3. After 201 iterations, the algorithm converged to the solution: $m_2^* = 2,948$ [Kg], $k_2^* = 23,897$ [kN/m], and $d_2^* = 116$ [kN s/m] with the control law given by

$$K^* = \begin{bmatrix}
190,499.5 & -187,499.3 & -1,254.6 & -25,891.8 & -5,650.8 & 4,386.9 \\
574,319.0 & -135,201.6 & -179,646.8 & -11,181.2 & -6,101.4 & -8,618.9 \\
447,891.3 & -21,021.9 & -163,367.5 & 4,487.7 & -4,548.0 & -20,177.4
\end{bmatrix}$$
This gain provides a control effort of $\sqrt{e[u^*u]} = 1.323,584$ [kN].

For this control gain $K^*$ and parameters $\alpha^*$, the achieved output performance is

$$C_p^y = \begin{pmatrix}
0.00017774233782 \\
0.0001999687426 \\
0.0000808336592 \\
0.29999795054617 \\
0.26623945460839 \\
0.14227944011144
\end{pmatrix}$$

For this case the binding constraints are $\varepsilon_2^z$, and $\varepsilon_3^z$.

Example 3. $(k_1,k_2,k_3,d_1,d_2,d_3,m_1,m_2,m_3)$. In this example, we follow the same steps as in the previous Example 2, but now all the parameters of the structure are optimized: $m_1$, $m_2$, $m_3$, $k_1$, $k_2$, $k_3$, $d_1$, $d_2$, and $d_3$. The same initial controller $K_0$ given in Eq. (14) is also used. After 75 iterations, the CASC algorithm gives the solution $e[u^*u] = 0$ (see Fig. 4). Thus, no active control is needed to achieve the prescribed performance, and the design is completely passive. The parameters are $m_1^* = 3.264$ [Kg], $m_2^* = 3.899$ [Kg], $m_3^* = 4.498$ [Kg], $k_1^* = 34.120$ [kN/m], $k_2^* = 27.923$ [kN/m], $k_3^* = 24.473$ [kN/m], $d_1^* = 133$ [kN m/s], $d_2^* = 116$ [kN m/s], and $d_3^* = 114$ [kN m/s].

### Table 3. Achieved Output Performance (Diagonal Entries of $C_p^y$)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Integrated</th>
<th>Active only</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>$k_2$, $d_2$</td>
<td>1.323,584</td>
</tr>
<tr>
<td>Example 2</td>
<td>$k_2$, $d_2$, $m_2$</td>
<td>$1.323,584$</td>
</tr>
<tr>
<td>Example 3</td>
<td>All parameters</td>
<td>0</td>
</tr>
</tbody>
</table>

### Table 4. Achieved Output Performance (Diagonal Entries of $C_p^y$)

<table>
<thead>
<tr>
<th>$e_2^z$ [m]</th>
<th>$e_3^z$ [m]</th>
<th>$e_2^z$ [m/s]</th>
<th>$e_3^z$ [m/s]</th>
<th>$e_2^z$ [m/s]</th>
<th>$e_3^z$ [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 4</td>
<td>4.00008</td>
<td>0.00009</td>
<td>0.00018566840635</td>
<td>$0.000019924123968$</td>
<td></td>
</tr>
<tr>
<td>Bound</td>
<td>0.00013</td>
<td>0.00013</td>
<td>0.00013</td>
<td>0.00120</td>
<td>0.00120</td>
</tr>
</tbody>
</table>

In this case, the achieved output performance is $C_p^y = \begin{pmatrix}
0.0018566840635 \\
0.00019924123968 \\
0.00009712073257 \\
0.29980689683670 \\
0.29985557567524 \\
0.16787420951997
\end{pmatrix}$.

Table 2 summarizes our findings. Using an integrated approach, no control effort is required to achieve $e_2^z \leq 0.0002$ [m] and $e_3^z \leq 0.3$ [m²/s²], $i = 1,2,3$. With feedback control fixed at the nominal parameters $(K_0)$ the control effort needed is $\sqrt{4,408,517}$ [kN].

For each of the previous three designs, the diagonal elements of the output covariance matrix are shown in Table 3.

Example 4. $(k_1,k_2,k_3,d_1,d_2,d_3,m_1,m_2,m_3)$. This example employs all the free parameters, but we change $\Omega$ by scaling by a factor $\mu$, that is $\Omega \mu$, in order to find the performance bound $\bar{\Omega}$ that represents the best performance that is achievable with only passive design. The active control energy $e[u^*u]$ as a function of the scaling factor $\mu$ is shown in Fig. 5. Thus, $\mu = 0.64$, i.e., $\bar{\Omega} = 0.64 \Omega$ represents the lowest bound on the output covariance for which the design is still completely passive.

For this bound $\bar{\Omega}$, the CASC algorithm converged after 264 iterations to the passive $(K_0 = 0$ and $e[u^*u] = 0$) solution: $m_1^* = 2.948$ [Kg], $m_2^* = 2.948$ [Kg], $m_3^* = 2.948$ [Kg], $k_1^* = 37.044$ [kN/m], $k_2^* = 28.155$ [kN/m], $k_3^* = 16.814$ [kN/m], $d_1^* = 134$ [kN m/s], $d_2^* = 116$ [kN m/s], and $d_3^* = 114$ [kN m/s]. For this system, the achieved output performance is given in Table 4.

Example 5. $(k_1,k_2,k_3,d_1,d_2,d_3,m_1,m_2,m_3)$. Now, we imposed a tighter upper bound, and obtain an active control law. We choose $\mu$ to be 0.4, thus $\Omega = 0.4 \Omega$, yielding performance 2.5 times better than the examples which used the performance criterion $C_p^y < \Omega$, for earthquakes intensity $W = 16$. Note that the performance $C_p^y$ simply scales with $W$. So the design we now discuss can also be interpreted as delivering the same performance bound as $\Omega$ in the presence of earthquakes of intensity $W = 2.5(16) = 40$. For this bound, the CASC algorithm converged after 142 iterations to the active solution: $m_1^* = 2.948$ [Kg], $m_2^* = 2.948$ [Kg], $m_3^* = 2.948$ [Kg], $k_1^* = 67.343$ [kN/m], $k_2^* = 46.289$ [kN/m], $k_3^* = 28.354$ [kN/m], $d_1^* = 134$ [kN m/s], $d_2^* = 116$ [kN m/s], and $d_3^* = 114$ [kN m/s] with the control law given by

### Table 5. Achieved Output Performance (Diagonal Entries of $C_p^y$)

<table>
<thead>
<tr>
<th>$e_2^z$ [m]</th>
<th>$e_3^z$ [m]</th>
<th>$e_2^z$ [m/s]</th>
<th>$e_3^z$ [m/s]</th>
<th>$e_2^z$ [m/s]</th>
<th>$e_3^z$ [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 5</td>
<td>0.000002</td>
<td>0.000002</td>
<td>0.000002</td>
<td>0.089999</td>
<td>0.08997</td>
</tr>
<tr>
<td>Bound</td>
<td>0.000006</td>
<td>0.000006</td>
<td>0.000006</td>
<td>0.090000</td>
<td>0.090000</td>
</tr>
</tbody>
</table>
Example 6. Now we simulate the passive system from Example 4, and the active system from Example 5, obtained by the CASC design, and the nominal system, when subjected to El Centro earthquake taken from Spencer Jr. et al. (1998) (not white noise). The results, presented in Fig. 6, show the superior performance of the passive system designed via the CASC algorithm over the nominal system (comparing only the passive systems). The active system shows a superior performance over the passive one. This

$$K^* = \begin{bmatrix} 307,824.1 & -390,158.3 & 80,694.4 & -64,017.3 & 3,997.3 & 3,588.3 \\ 827,567.6 & -216,198.7 & -125,413.8 & 3,072.2 & -47,713.6 & 3,965.2 \\ 190,682.3 & 165,905.7 & -122,324.2 & 4,226.7 & 3,628.9 & -42,525.5 \end{bmatrix}$$

This gain provides a control effort of $\sqrt{e'[u^*u]} = \sqrt{1,536,541}$ [kN]. The achieved output performance is given in Table 5.

Fig. 6. Response of nominal system, passive system from Example 4, and active system from Example 5 due to El Centro earthquake
was expected since we imposed a tighter output covariance upper bound.

Conclusion

This paper demonstrates the benefits of simultaneously designing the structure and the controller for civil structures. This opens the doors to the use of control methods to design structures, even when no control is intended. Since the control methods allow bounds to be placed on the dynamic response, this should be a welcome improvement in structure design. The algorithm is proposed in the LMI framework, for which very efficient interior point methods are available. The design allows changes in any parameters that appear affinely in the mass, the damping, and the stiffness matrices. The nonconvex mixed structure/control problem is solved by a sequence of convex subproblems with the help of potential convexifying functions. The performance criteria in the design was a bound on the output covariance of the closed loop system, but the methodology can incorporate many other convex criteria. Simulations show that the proposed algorithm possesses nice convergence properties. The paper improves the techniques available in the literature in the sense that the methodology is completely in the LMI framework, having no need to solve a constrained quadratic optimization problem; the technique allows parameters in the mass matrix to be optimized; and the proposed algorithm does not require that the Lyapunov matrix (which provides the control gain) be fixed in the structure design step.

For a specified performance bound, the proposed CASC algorithm determines whether active feedback control is necessary or not. The problem minimizes the control energy required to achieve a specified bound $\Omega$ on the response. If the specified bound $\Omega$ is large enough the algorithm produces a passive design (indicated by zero control energy in the optimized solution). If the required performance bound $\Omega$ is small enough, no passive solution exists, and the algorithm produces a controller with minimal control energy to achieve the required performance $\Omega$.

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References


