Gain-Scheduled $\mathscr{H}_\infty$-Control for Discrete-Time Polytopic LPV Systems Using Homogeneous Polynomially Parameter-Dependent Lyapunov Functions

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Abstract: This paper presents $\mathscr{H}_\infty$ performance analysis and control synthesis for discrete-time linear systems with time-varying parameters. The parameters are assumed to vary inside a polytope and have known bounds on their rate of variation. The geometric properties of the polytopic domain are exploited to derive parameter-dependent linear matrix inequality conditions that consider the bounds on the rate of variation of the parameters. A systematic procedure is proposed to construct a family of finite-dimensional relaxations based on Pólya's Theorem and a homogeneous polynomially parameter-dependent parameterization of arbitrary degree for the Lyapunov matrix. A numerical example illustrates the proposed approach.

Keywords: Time-varying systems ; Discrete-time systems ; $\mathscr{H}_\infty$-control ; Feedback control ; Lyapunov function

1. INTRODUCTION

Performance analysis and control synthesis for linear parameter-varying (LPV) systems have received a lot of attention from the control community. This stems from the fact that LPV models are useful to describe the dynamics of linear systems with time-varying parameters as well as to represent nonlinear systems in terms of a family of linear models (see Rugh and Shamma [2000]). In the LPV control framework, the scheduling parameters that govern the variation of the system dynamics are usually unknown in advance, but supposed to be measured or estimated in real-time (see Shamma and Athans [1991]). There is a continuing effort towards the design of LPV controllers, scheduled as a function of the varying parameters, to achieve higher performance while still guaranteeing stability for all possible parameter variations (see, for instance, Apkarian and Adams [1998], Rugh and Shamma [2000], Apkarian et al. [1995], Packard [1994], Leith and Leithead [2000], Shamma and Athans [1992], Scherer [2001]).

In the literature, several analysis and synthesis results for LPV systems have been proposed based on different types of Lyapunov functions that are able to guarantee stability and performance. The appeal of Lyapunov theory comes from the fact that it allows to recast many analysis and synthesis problems as linear matrix inequality (LMI) optimization problems (see Boyd et al. [1994]). For LPV systems, the resulting parameter-dependent LMI conditions need to be satisfied for the entire parameter space and, consequently, these LMI problems are infinite-dimensional. To arrive at a finite-dimensional set of LMI conditions, the choice of the parameterization (or the structure) of the Lyapunov matrix is essential.

Many of the Lyapunov approaches (see, for example, Khargonekar et al. [1993], Bernussou et al. [1989], Montagner et al. [2005b]) use the notion of quadratic stability where the Lyapunov matrix is constant. This yields a finite set of LMIs that are usually conservative for practical applications, since it allows arbitrarily fast variation of the scheduling parameters. To alleviate some of the conservatism associated with the quadratic stability-based approaches, many works have proposed the use of parameter-dependent Lyapunov functions: for time-varying systems, piecewise Lyapunov matrices are considered by, amongst others, Leite and Peres [2004] and Amato et al. [2005], affine and polytopic structures are used in, for instance, Daafouz and Bernussou [2001], Montagner et al. [2005a], Oliveira and Peres [2008], De Caigny et al. [2008a,b] and homogeneous polynomially parameter-dependent (HPPD) matrices in, for example, Montagner et al. [2006].

The aim of this work is to extend the recent results of De Caigny et al. [2008a,b], by constructing a family of finite-dimensional LMI relaxations based on homogenous polynomially parameter-dependent Lyapunov (HPPDL) functions and real algebraic properties, for the parameter-dependent LMIs associated with $\mathscr{H}_\infty$ performance analysis and control synthesis for discrete-time linear systems with time-varying parameters belonging to a polytope with a prescribed bound on the rate of variation. In this way, the analysis results in this paper extend the results presented in Oliveira and Peres [2007] for uncertain time-invariant parameters to the case of discrete-time systems with time-varying parameters. The control synthesis is restricted to the case where the measurement equation is unaffected by the control inputs, the exogenous disturbance inputs and the scheduling parameter.

The paper is organized as follows. Section 2 provides preliminary material concerning the modeling of the polytopic domain. Sections 3 and 4 present $\mathscr{H}_\infty$ performance analysis and static output feedback control synthesis using HPPDL functions and Pólya's relaxations. A numerical example is presented in Section 5 that shows the benefits of the proposed approach. The conclusions are presented in Section 6.

2. PRELIMINARIES

Consider the discrete-time polytopic linear time-varying system

$$H := \begin{cases} x[t_{k+1}] = A(\alpha[t_k])\, x[t_k] + B_w(\alpha[t_k])\, w[t_k] + B_u(\alpha[t_k])\, u[t_k], \\ z[t_k] = C_z(\alpha[t_k])\, x[t_k] + D_w(\alpha[t_k])\, w[t_k] + D_u(\alpha[t_k])\, u[t_k], \end{cases} \tag{1}$$

where $x[t_k] \in \mathbb{R}^n$ is the state, $w[t_k] \in \mathbb{R}^r$ the exogenous input, $u[t_k] \in \mathbb{R}^m$ the control input and $z[t_k] \in \mathbb{R}^p$ the system output. The system matrices $A(\alpha[t_k]) \in \mathbb{R}^{n\times n}$, $B_w(\alpha[t_k]) \in \mathbb{R}^{n\times r}$, $B_u(\alpha[t_k]) \in \mathbb{R}^{n\times m}$, $C_z(\alpha[t_k]) \in \mathbb{R}^{p\times n}$, $D_w(\alpha[t_k]) \in \mathbb{R}^{p\times r}$ and $D_u(\alpha[t_k]) \in \mathbb{R}^{p\times m}$ belong to the polytope

$$\mathscr{D} = \Big\{ (A,B_w,B_u,C_z,D_w,D_u)(\alpha[t_k]) : \\ (A,B_w,B_u,C_z,D_w,D_u)(\alpha[t_k]) \\ = \sum_{i=1}^{N} \alpha_i[t_k]\,(A_i,B_{w,i},B_{u,i},C_{z,i},D_{w,i},D_{u,i})\,, \alpha[t_k] \in \Lambda_N \Big\},$$

where, for all $k \geq 0$, the vector of time-varying parameters $\alpha[t_k]$ belongs to the unit simplex

$$\Lambda_N = \Big\{ \xi \in \mathbb{R}^N : \sum_{i=1}^{N} \xi_i = 1, \xi_i \geq 0, i = 1,\dots,N \Big\}. \tag{2}$$

The rate of variation of the parameters

$$\Delta\alpha_i[t_k] = \alpha_i[t_{k+1}] - \alpha_i[t_k], \quad i = 1,\dots,N, \tag{3}$$

is assumed to be limited by an *a priori* known bound b such that

$$-b\alpha_i[t_k] \leq \Delta\alpha_i[t_k] \leq b(1-\alpha_i[t_k]), \quad i = 1,\dots,N, \tag{4}$$

with $b \in \mathbb{R},\ b \in [0,1]$. As recently discussed in Oliveira and Peres [2008], less conservative analysis and synthesis results can be obtained by explicitly taking into account that $\Delta\alpha_i[t_k]$ satisfies (4). Following Oliveira and Peres [2008], the space where $\Delta\alpha[t_k]$ can assume values can be modeled by the set

$$\Gamma_b = \Big\{ \delta \in \mathbb{R}^N : \delta \in co\left\{h^1,\dots,h^N\right\}, \\ \sum_{i=1}^{N} h_i^j = 0, \quad h^j \in \mathbb{R}^N, \quad j = 1,\dots,N \Big\},$$

where $co\{X\}$ denotes the convex hull of a set X. The first step to construct the vectors h^j is to observe that (2) and (3) yield $\sum_{i=1}^{N} \Delta\alpha_i[t_k] = 0$. Solving this equality under the extreme values of (4), yields the following vectors (solutions) h^j (depending on both b and $\alpha[t_k]$)

$$\left[h^1 \cdots h^N\right] = b \begin{bmatrix} 1-\alpha_1[t_k] & -\alpha_1[t_k] & \dots & -\alpha_1[t_k] \\ -\alpha_2[t_k] & 1-\alpha_2[t_k] & \dots & -\alpha_2[t_k] \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_N[t_k] & -\alpha_N[t_k] & \dots & 1-\alpha_N[t_k] \end{bmatrix}.$$

By taking the convex combination of the N columns h^j using $\beta[t_k] \in \Lambda_N$, the following expression is obtained

$$\Delta\alpha_j[t_k] = b(\beta_j[t_k] - \alpha_j[t_k]). \tag{5}$$

This characterization of $\Delta\alpha_j[t_k]$ will be exploited in Sections 3 and 4 to derive LMI conditions for $\mathscr{H}_\infty$ performance analysis and control synthesis for system (1), based on HPPDL functions and Pólya's relaxations.

Remark 1. Note that the proposed approach can also be used when the bound on the rate of parameter variation is different for each scheduling parameter, that is, $-b_i \leq \Delta\alpha_i[tk] \leq b_i$, for $i = 1,\dots,N$. However, in this case, the set Γ_b where $\Delta\alpha[t_k]$ can assume values cannot be constructed in a general systematic way and the algorithms to construct the finite sets of LMIs are more complex.

2.1 Notation

The symbol $'$ denotes the transpose. The ℓ_2^n space of square-summable sequences on the set of nonnegative integers $\mathbb{Z}_+$ is given by

$$\ell_2^n \triangleq \left\{ f : \mathbb{Z}_+ \to \mathbb{R}^n \;\Big|\; \sum_{k=0}^{\infty} f[t_k]' f[t_k] < \infty \right\}.$$

The 2-norm is defined as $\|f[t_k]\|_2^2 = \sum_{k=0}^{\infty} f[t_k]' f[t_k]$. The identity matrix of size $r \times r$ is denoted as I_r. The notation $\mathbf{0}_{n\times m}$ indicates an $n \times m$ matrix of zeros. For reasons of compactness, the scheduling parameter α at time instant k is denoted by $\alpha = \alpha[t_k]$ and at time instant $k+1$ by $\alpha^+ = \alpha[t_{k+1}]$. Whenever there is no ambiguity, the sum of the M elements of a vector $a \in \mathbb{R}^M$ is denoted by $\sum a = \sum_{i=1}^{M} a_i$.

The set of N-tuples obtained as all possible combinations of nonnegative integers $k_i \in \mathbb{Z}_+$ such that $k_1 + k_2 + \cdots + k_N = g$ is denoted as $\mathscr{K}(g)$. The number of elements in $\mathscr{K}(g)$ is given by

$$J(g) = \frac{(N+g-1)!}{g!(N-1)!}.$$

For N-tuples k and ℓ, one writes $k \succeq \ell$ if $k_i \geq \ell_i$, $\forall i$. Operations of summation $k+\ell$ and subtraction $k-\ell$ (if $k \succeq \ell$) are defined elementwise. The following notation for multivariable homogeneous polynomials is used. An HPPD matrix of degree g is denoted by

$$P_g(\alpha) = \sum_{k \in \mathscr{K}(g)} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_N^{k_N} P_k,$$

with $\alpha \in \Lambda_N$ and $k_i \in \mathbb{Z}_+$, for $i = 1,\dots,N$, represent the monomials and $P_k \in \mathbb{R}^{n\times n}$, $\forall k \in \mathscr{K}(g)$ are the matrix-valued coefficients. The following short notation is used for the monomial $\alpha^k = \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_N^{k_N}$. Finally, consider the following definitions for the N-tuple

$$e_i = 0 \cdots 0 \underbrace{1}_{\text{i-th}} 0 \dots 0,$$

and the value $\pi(k) \triangleq (k_1!)(k_2!)\cdots(k_N!) = \prod_{i=1}^{N}(k_i!)$.

3. GUARANTEED $\mathscr{H}_\infty$ PERFORMANCE

The aim of this section is to characterize through a finite set of LMIs an upper bound γ on the $\mathscr{H}_\infty$ performance of system

(1) in open-loop, such that for any input $w[t_k] \in \ell_2^r$, the system output $z[t_k] \in \ell_2^p$ satisfies

$$\|z[t_k]\|_2 < \gamma \, \|w[t_k]\|_2, \qquad 0 < \gamma,$$

for any possible variation of the parameter $\alpha \in \Lambda_N$ with prescribed bound b on its rate of variation.

Based on the bounded real lemma, an upper bound for the $\mathscr{H}_\infty$ performance can be characterized by the following theorem, as shown in de Souza et al. [2006].

Theorem 2. Consider the system H, given by (1). If there exist parameter-dependent matrices $G(\alpha)$ and $P(\alpha) = P(\alpha)' > 0$ such that $\Theta(\alpha) > 0$, for all $\alpha \in \Lambda_N$, with

$$\Theta(\alpha) = \begin{bmatrix} P(\alpha^+) & A(\alpha)G(\alpha) & B_w(\alpha) & \mathbf{0}_{n\times p} \\ \star & G(\alpha)+G(\alpha)'-P(\alpha) & \mathbf{0}_{n\times r} & G(\alpha)'C_z(\alpha)' \\ \star & \star & \gamma I_r & D_w(\alpha)' \\ \star & \star & \star & \gamma I_p \end{bmatrix} \tag{6}$$

then the system H is exponentially stable and

$$||H||_\infty \leq \inf_{P(\alpha),G(\alpha),\gamma} \gamma.$$

It is worth to emphasize that the condition $\Theta(\alpha) > 0$ of Theorem 2, which consists in evaluating the parameter-dependent LMI for all α in the unit simplex Λ_N, leads to an infinite dimensional problem. However, by parameterizing the Lyapunov matrix $P(\alpha)$ and the slack variable $G(\alpha)$ as

$$\begin{aligned} P(\alpha) = P_{\bar{g}}(\alpha) &= \sum_{k\in\mathscr{K}(\bar{g})} \alpha^k P_j, \\ G(\alpha) = G_{\bar{g}}(\alpha) &= \sum_{k\in\mathscr{K}(\bar{g})} \alpha^k G_j, \end{aligned} \tag{7}$$

a finite set of LMIs in terms of the vertices of the polytope $\mathscr{D}$ can be obtained. Moreover, when an HPPD solution (7) exists for (6), a sequence of relaxations can be constructed based on an extension of Pólya's Theorem to the case of matrix-valued coefficients, as introduced in Scherer [2005]. The relaxations are based on the fact that if $P_{\bar{g}}(\alpha) = P_{\bar{g}}(\alpha)' > 0$ and $G_{\bar{g}}(\alpha)$ are feasible solutions for $\Theta(\alpha) > 0$, with $\Theta(\alpha)$ given by (6), then

$$P_g(\alpha) = \left(\sum_{i=1}^{N} \alpha_i\right)^f P_{\bar{g}}(\alpha) > 0 \text{ and } G_g(\alpha) = \left(\sum_{i=1}^{N} \alpha_i\right)^f G_{\bar{g}}(\alpha), \tag{8}$$

with $g = f + \bar{g}$, are feasible solutions for

$$\left(\sum_{i=1}^{N} \alpha_i\right)^d \Theta(\alpha) > 0, \tag{9}$$

since $\sum_{i=1}^{N} \alpha_i = 1$. The following theorem provides a finite set of sufficient LMIs that guarantee (9) using (8).

Theorem 3. Consider system H, given by (1). Let f, d and $\bar{g}$ be given. If there exist matrices $G_k \in \mathbb{R}^{n\times n}$ and $P_k = P_k' \in \mathbb{R}^{n\times n}$ such that the LMIs

$$T_\ell = \sum_{\substack{k\in\mathscr{K}(\bar{g}) \\ \ell\succeq k}} \frac{f!}{\pi(\ell-k)} P_k > 0, \tag{10}$$

hold, $\forall \ell \in \mathscr{K}(f+\bar{g})$, and the LMIs

$$R_{\ell j} = \begin{bmatrix} R_{11,\ell j} & R_{12,\ell j} & R_{13,\ell j} & \mathbf{0}_{n\times p} \\ \star & R_{22,\ell j} & \mathbf{0}_{n\times r} & R_{24,\ell j} \\ \star & \star & R_{33,\ell j} & R_{34,\ell j} \\ \star & \star & \star & R_{44,\ell j} \end{bmatrix} > 0, \tag{11}$$

hold, $\forall \ell \in \mathscr{K}(f+\bar{g}+d+1)$, $\forall j \in \mathscr{K}(f+\bar{g})$, where

$$R_{11,\ell j} = \sum_{\substack{h\in\mathscr{K}(f+\bar{g}+1) \\ \ell\succeq h}} \sum_{k=0}^{f} \sum_{i=0}^{\bar{g}} \sum_{\substack{m_1\in\mathscr{K}(k+i) \\ h\succeq m_1}} \sum_{\substack{m_2\in\mathscr{K}(f+\bar{g}-k-i) \\ j\succeq m_2}} \sum_{\substack{t_1\in\mathscr{K}(i) \\ m_1\succeq t_1}} \sum_{\substack{t_2\in\mathscr{K}(\bar{g}-i) \\ m_2\succeq t_2}} \frac{d!(f+\bar{g}+1-k-i)!(k+i)!f!\pi(t_1+t_2)\,(1-b)^{\Sigma m_1} b^{\Sigma m_2}}{\pi(\ell-h)\pi(h-m_1)\pi(j-m_2)\pi(m_1-t_1)\pi(m_2-t_2)\pi(t_1)\pi(t_2)} P_{t_1+t_2}, \tag{12}$$

$$R_{12,\ell j} = \sum_{\substack{h\in\mathscr{K}(\bar{g}+1) \\ \ell\succeq h}} \sum_{\substack{i=1 \\ h_i>0}}^{N} \frac{(f+d)!(f+\bar{g})!}{\pi(\ell-h)\pi(j)} A_i G_{h-e_i}, \tag{13}$$

$$R_{13,\ell j} = \sum_{\substack{i=1 \\ \ell_i>0}}^{N} \frac{(f+\bar{g})!(f+\bar{g}+d)!}{\pi(j)\pi(\ell-e_i)} B_{w,i}, \tag{14}$$

$$R_{22,\ell j} = \sum_{\substack{h\in\mathscr{K}(\bar{g}) \\ \ell\succeq h}} \frac{(f+\bar{g})!(f+d+1)!}{\pi(j)\pi(\ell-h)} \left(G_h + G_h' - P_h\right), \tag{15}$$

$$R_{24,\ell j} = \sum_{\substack{h\in\mathscr{K}(\bar{g}+1) \\ \ell\succeq h}} \sum_{\substack{i=1 \\ h_i>0}}^{N} \frac{(f+d)!(f+\bar{g})!}{\pi(\ell-h)\pi(j)} G_{h-e_i}' C_{z,i}', \tag{16}$$

$$R_{33,\ell j} = \frac{(f+\bar{g}+d+1)!(f+\bar{g})!}{\pi(\ell)\pi(j)} \gamma I_r, \tag{17}$$

$$R_{34,\ell j} = \sum_{\substack{i=1 \\ \ell_i>0}}^{N} \frac{(f+\bar{g})!(f+\bar{g}+d)!}{\pi(j)\pi(\ell-e_i)} D_{w,i}', \tag{18}$$

$$R_{44,\ell j} = \frac{(f+\bar{g}+d+1)!(f+\bar{g})!}{\pi(\ell)\pi(j)} \gamma I_p, \tag{19}$$

then the system H is exponentially stable and

$$||H||_\infty \leq \min_{P_k,G_k,\gamma} \gamma.$$

Proof.

Using (8) in (9), it follows that the maximal polynomial degree of (9) in α is equal to $\bar{g}+f+d+1$ (in block $(\sum\alpha)^d A(\alpha)G(\alpha)$). Using (5), it follows that

$$P(\alpha^+) = P(\alpha+\Delta\alpha) = P((1-b)\alpha + b\beta). \tag{20}$$

Therefore, the maximal polynomial degree of (9) in β is $\bar{g}+f$ (in block $(\sum\alpha)^d P(\alpha^+)$). Consequently, a homogenization is necessary for all blocks in (9), such that they all have polynomial degree $\bar{g}+f+d+1$ in α and polynomial degree $\bar{g}+f$ in β. This homogenization leads to the LMI blocks (12)–(19) and therefore, multiplying (11) with $\alpha^\ell \beta^j$ and summing for $\ell \in \mathscr{K}(\bar{g}+f+d+1)$ and $j \in \mathscr{K}(\bar{g}+f)$ yields (9). Analogously, multiplying (10) with α^ℓ and summing for $\ell \in \mathscr{K}(f+\bar{g})$ guarantees $P_g(\alpha) = (\sum\alpha)^f P_{\bar{g}}(\alpha) > 0$.

4. STATIC OUTPUT FEEDBACK SYNTHESIS

In this section, the analysis result presented in Theorem 3 is extended to provide a finite set of LMI conditions for the synthesis of a gain-scheduled $\mathscr{H}_\infty$ static output feedback controller for system (1). It is assumed that the first q states of the system can be measured in real-time for feedback without corruption by the exogenous input $w[t_k]$ or the control input $u[t_k]$, that is,

$$y[t_k] = C_y\, x[t_k], \quad C_y = [I_q \;\; \mathbf{0}_{q\times n-q}]\,. \tag{21}$$

If this is not the case, one can use a similarity transformation as proposed in Kailath [1980], whenever the output matrix C_y is not affected by the time-varying parameter.

The goal is to provide a parameter-dependent control law

$$u[t_k] = K(\alpha) y[t_k], \qquad \text{with} \quad K(\alpha) \in \mathbb{R}^{m\times q},$$

such that the closed-loop system

$$\begin{cases} x[t_{k+1}] = A_{cl}(\alpha)\, x[t_k] + B_w(\alpha)\, w[t_k], \\ z[t_k] = C_{cl}(\alpha)\, x[t_k] + D_w(\alpha)\, w[t_k], \end{cases} \tag{22}$$

with

$$A_{cl}(\alpha) = A(\alpha) + B_u(\alpha)K(\alpha)C_y,$$
$$C_{cl}(\alpha) = C_z(\alpha) + D_u(\alpha)K(\alpha)C_y,$$

is exponentially stable with a guaranteed $\mathscr{H}_\infty$ performance for all possible variation of the parameter $\alpha[t_k] \in \Lambda_N$. A solution to this $\mathscr{H}_\infty$ static output feedback design problem, in terms of a finite set of LMIs, is provided by the next theorem.

Theorem 4. Consider system H, given by (1). Let f, d and $\bar{g}$ be given. If there exist matrices $G_{1,k} \in \mathbb{R}^{q\times q}$, $G_{2,k} \in \mathbb{R}^{n-q\times q}$, $G_{3,k} \in \mathbb{R}^{n-q\times n-q}$, $Z_{1,k} \in \mathbb{R}^{m\times q}$ and $P_k = P_k' \in \mathbb{R}^{n\times n}$ such that the LMIs (10) hold, $\forall \ell \in \mathscr{K}(f+\bar{g})$, and the LMIs

$$S_{\ell j} = \begin{bmatrix} R_{11,\ell j} & S_{12,\ell j} & R_{13,\ell j} & \mathbf{0}_{n\times p} \\ \star & R_{22,\ell j} & \mathbf{0}_{n\times r} & S_{24,\ell j} \\ \star & \star & R_{33,\ell j} & R_{34,\ell j} \\ \star & \star & \star & R_{44,\ell j} \end{bmatrix} > 0, \tag{23}$$

hold, $\forall \ell \in \mathscr{K}(f+\bar{g}+d+1)$, $\forall j \in \mathscr{K}(f+\bar{g})$, with $R_{11,\ell j}$, $R_{13,\ell j}$, $R_{22,\ell j}$, $R_{33,\ell j}$, $R_{34,\ell j}$ and $R_{44,\ell j}$, respectively given by (12), (14), (15), (17), (18) and (19), and

$$S_{12,\ell j} = \sum_{\substack{h\in\mathscr{K}(\bar{g}+1)\\ \ell\succeq h}} \sum_{\substack{i=1\\ h_i>0}}^{N} \frac{(f+d)!(f+\bar{g})!}{\pi(\ell-h)\pi(j)} \left(A_i G_{h-e_i} + B_{u,i} Z_{h-e_i}\right), \tag{24}$$

$$S_{24,\ell j} = \sum_{\substack{h\in\mathscr{K}(\bar{g}+1)\\ \ell\succeq h}} \sum_{\substack{i=1\\ h_i>0}}^{N} \frac{(f+d)!(f+\bar{g})!}{\pi(\ell-h)\pi(j)} \left(C_{z,i} G_{h-e_i} + D_{u,i} Z_{h-e_i}\right)', \tag{25}$$

where

$$G_k = \begin{bmatrix} G_{1,k} & \mathbf{0}_{q\times n-q} \\ G_{2,k} & G_{3,k} \end{bmatrix} \quad \text{and} \quad Z_k = \begin{bmatrix} Z_{1,k} & \mathbf{0}_{m\times n-q} \end{bmatrix} \tag{26}$$

then the parameter-dependent static output feedback gain

$$K(\alpha) = \hat{Z}_{\bar{g}}(\alpha)\hat{G}_{\bar{g}}(\alpha)^{-1}, \tag{27}$$

with

$$\hat{Z}_{\bar{g}}(\alpha) = \sum_{k\in\mathscr{K}(\bar{g})} \alpha^k Z_{1,k}, \quad \text{and} \quad \hat{G}_{\bar{g}}(\alpha) = \sum_{k\in\mathscr{K}(\bar{g})} \alpha^k G_{1,k}, \tag{28}$$

stabilizes the system H for all $\alpha \in \Lambda_N$ and $\Delta\alpha \in \Gamma_b$, with a guaranteed closed-loop $\mathscr{H}_\infty$ performance bounded by γ^*, given by

$$\gamma^* = \min_{P_k, G_{1,k}, G_{2,k}, G_{3,k}, Z_{1,k}, \gamma} \gamma.$$

Proof.

Multiplying (23) with $\alpha^\ell \beta^j$ and summing for $\ell \in \mathscr{K}(\bar{g}+f+d+1)$ and $j \in \mathscr{K}(\bar{g}+f)$ yields $\Psi(\alpha) > 0$ with

$$\Psi(\alpha) = \begin{bmatrix} P_g(\alpha^+) & A(\alpha)G_g(\alpha) + B_u(\alpha)Z_g(\alpha) & B_w(\alpha) & \mathbf{0}_{n\times p} \\ \star & G_g(\alpha) + G_g(\alpha)' - P_g(\alpha) & \mathbf{0}_{n\times r} & G_g(\alpha)'C_z(\alpha)' + Z_g(\alpha)'D_u(\alpha)' \\ \star & \star & \gamma I_r & D_w(\alpha)' \\ \star & \star & \star & \gamma I_p \end{bmatrix}.$$

Using (26) and (27) and considering the specific structure (21) for C_y, the LMI $\Psi(\alpha) > 0$ can be written as

$$\begin{bmatrix} P_g(\alpha^+) & A_{cl}(\alpha)G_g(\alpha) & B_w(\alpha) & \mathbf{0}_{n\times p} \\ \star & G_g(\alpha) + G_g(\alpha)' - P_g(\alpha) & \mathbf{0}_{n\times r} & G_g(\alpha)'C_{cl}(\alpha)' \\ \star & \star & \gamma I_r & D_w(\alpha)' \\ \star & \star & \star & \gamma I_p \end{bmatrix} > 0.$$

Therefore, due to Theorem 2, feasibility of the set of LMIs (23) ensures that the closed-loop system (22) is exponentially stable with a guaranteed $\mathscr{H}_\infty$ performance bounded by γ^*.

Synthesis conditions for a robust $\mathscr{H}_\infty$ static output feedback controller

$$u[t_k] = K\, y[t_k]$$

can be derived from Theorem 4 by enforcing the slack variables $\hat{G}_{\bar{g}}(\alpha)$ and $\hat{Z}_{\bar{g}}(\alpha)$ in (28) to be parameter-independent and by applying the following homogenization

$$\hat{G}_{\bar{g}}(\alpha) = G_{11} = \left(\sum \alpha\right)^{\bar{g}} G_{11} = \sum_{k\in\mathscr{K}(\bar{g})} \alpha^k \frac{\bar{g}!}{\pi(k)} G_{11},$$

$$\hat{Z}_{\bar{g}}(\alpha) = Z_{11} = \left(\sum \alpha\right)^{\bar{g}} Z_{11} = \sum_{k\in\mathscr{K}(\bar{g})} \alpha^k \frac{\bar{g}!}{\pi(k)} Z_{11}.$$

This is the context of Corollary 5.

Corollary 5. Consider system H, given by (1). Let f, d and $\bar{g}$ be given. If there exist matrices $G_{11} \in \mathbb{R}^{q\times q}$ and $Z_{11} \in \mathbb{R}^{m\times q}$, and matrices $G_{2,k} \in \mathbb{R}^{n-q\times q}$, $G_{3,k} \in \mathbb{R}^{n-q\times n-q}$ and $P_k = P_k' \in \mathbb{R}^{n\times n}$ such that the LMIs (10) hold, $\forall \ell \in \mathscr{K}(f+\bar{g})$, and the LMIs (23) hold, $\forall \ell \in \mathscr{K}(f+\bar{g}+d+1)$, $\forall j \in \mathscr{K}(f+\bar{g})$, with

$$G_k = \begin{bmatrix} \frac{\bar{g}!}{\pi(k)} G_{11} & \mathbf{0}_{q\times n-q} \\ G_{2,k} & G_{3,k} \end{bmatrix}, \; Z_k = \begin{bmatrix} \frac{\bar{g}!}{\pi(k)} Z_{11} & \mathbf{0}_{m\times n-q} \end{bmatrix}, \tag{29}$$

then the parameter-independent static output feedback gain

$$K = Z_{11} G_{11}^{-1},$$

stabilizes the system H for all $\alpha \in \Lambda_N$ and $\Delta\alpha \in \Gamma_b$, with a guaranteed closed-loop $\mathscr{H}_\infty$ performance bounded by γ^*, given by

$$\gamma^* = \min_{P_k, G_{11}, G_{21}, G_{22}, Z_{11}, \gamma} \gamma.$$

In case all states are available for feedback, the conditions of Theorem 4 and Corollary 5 can be used to design a gain-scheduled or a robust state feedback controller.

4.1 Discussion

It is important to note that several results in the literature can be recovered from the results of Theorems 3 and 4. First, by enforcing the bound on the rate of parameter variation to be $b = 0$, system (1) becomes a linear time-invariant system with uncertain parameters in a polytope. In this case, the analysis conditions of Oliveira and Peres [2007] can be recovered from the conditions of Theorem 3. Second, in De Caigny et al. [2008a,b], analysis and synthesis is considered based on affine Lyapunov matrices. These results can be recovered from Theorems 3 and 4 by choosing $\bar{g} = 1$, $f = 0$ and $d = 0$. Third, when the bound on the rate of parameter variation is chosen to be $b = 1$, for $\bar{g} = 1$, $f = 0$ and $d = 0$, the results from Montagner et al. [2005a] are recovered. On the other hand, if the bound is chosen to be $b = 0$, for $\bar{g} = 1$, $f = 0$ and $d = 0$, and the slack variables G_k are taken constant, that is, $G_k = G$, the results of Theorem 4 from de Oliveira et al. [2002] are recovered.

5. NUMERICAL EXAMPLE

Consider the polytopic time-varying system (1) for $n = 3$ and $N = 2$ with the following system matrices:

$$\left[\, A_1 \,\vdots\, A_2 \,\right] = \mu \left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 2 & -1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & -2 & -1 \end{array}\right], B_{w,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$B_{w,2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B_{u,i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_{z,i} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T, D_{u,i} = D_{w,i} = 0,$$

$i = 1,2$, with $\mu \in \mathbb{R}$ a given nonnegative scalar. The goal is to determine the maximum bound $b_{\max}$ on the rate of variation as a function of μ such that the system can be stabilized by an $\mathscr{H}_\infty$ static output feedback controller. Both gain-scheduled (GS) and robust (R) controllers are designed for two different cases of the measurement equation $y[t_k] = C_y x[t_k]$:

(1) $q = 1$: first state measured;
(2) $q = 2$: first two states measured.

Moreover, four different relaxation cases are considered.

(1) $\bar{g} = 1$, $f = 0$, $d = 0$: an affine Lyapunov function, without Pólya's relaxation;
(2) $\bar{g} = 3$, $f = 0$, $d = 0$: an HPPDL function of degree 3, without Pólya's relaxation;
(3) $\bar{g} = 1$, $f = 3$, $d = 3$: an affine Lyapunov function, with Pólya's relaxations of degree 3;
(4) $\bar{g} = 4$, $f = 1$, $d = 1$: an HPPDL function of degree 4, with Pólya's relaxations of degree 1.

The different control design cases are indicated as $R_{i,(q=j)}$ and $GS_{i,(q=j)}$, using j to indicate the measurement case $j = \{1,2\}$ and i to indicate the relaxation case $i = \{1,2,3,4\}$.

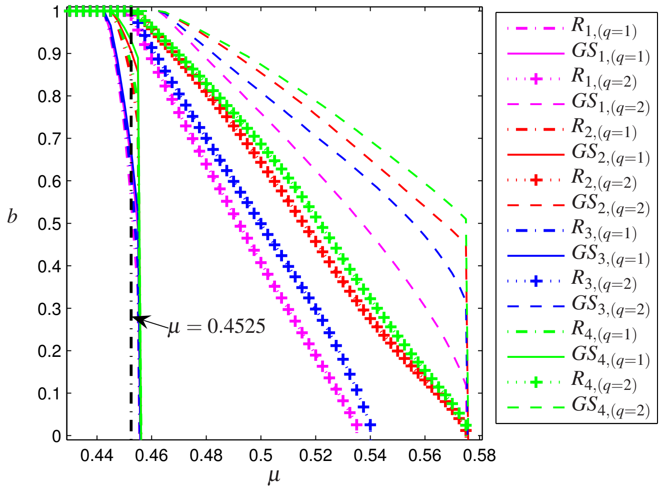

Fig. 1. Maximal bound $b_{\max}$ on the rate of parameter variation as a function of the scalar μ.

Figure 1 shows the results for these 16 control designs. Several conclusions can be drawn. First, it is clear that for low values of μ, all designs result in controllers that allow the parameters to vary arbitrarily fast in the unit simplex since $b_{\max} = 1$. However, as μ increases, the maximal allowed bound $b_{\max}$ becomes smaller. Obviously, this occurs first for the robust cases $R_{i,(q=1)}$, since they are the most restrictive control designs. Second, the curves for $R_{i,(q=2)}$ (resp. $GS_{i,(q=2)}$) are on the right of $R_{i,(q=1)}$ (resp. $GS_{i,(q=1)}$), which means that if more states can be measured, stabilizing controllers can be designed for higher values of μ. Third, since the gain-scheduled controllers are less restrictive than the robust controllers, the curve associated with $GS_{i,(q=j)}$ is always on the right of the curve of the corresponding robust case $R_{i,(q=j)}$).

To check the achieved performance of the different control designs, μ is now fixed to be $\mu = 0.4525$. Figure 2 shows the achieved upper bound γ on the closed-loop $\mathscr{H}_\infty$ performance as a function of the allowed bound $0 \le b \le 1$ on the rate of variation. For all control designs, it is clear from Figure 2, that as the bound b increases, the performance becomes worse since the upper bound γ increases. In the robust cases $R_{i,(q=j)}, \forall i, \forall j$,

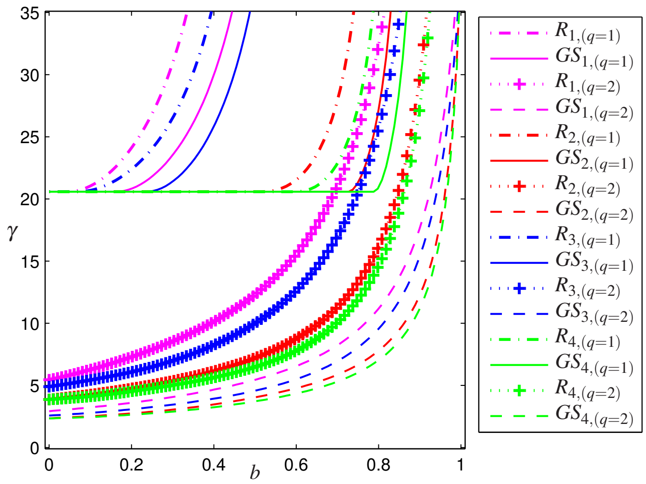

Fig. 2. Guaranteed upper bound γ on the $\mathscr{H}_\infty$ cost.

and the gain-scheduled cases $GS_{i,(q=1)}, \forall i$, the upper bound γ increases very fast as the value of the bound b increases. This can be expected since Figure 1 shows that if $\mu = 0.4525$ (black dash-dotted line) the synthesis conditions for these cases become infeasible for values of $b < 1$. Figure 2 also shows that the different relaxation cases result in different achieved performance. The relaxation case 4 (dashed curves), using HPPD matrices of degree 4 in combination with Pólya's relaxations of degree 1, yields the best performance.

Finally, the numerical complexity of the different relaxation cases is compared, by evaluating the number of scalar decision variables V and the number of LMI rows R. For both robust and gain-scheduled cases, the number V_P of scalar variables in the Lyapunov matrices P_k, is given by

$$V_P = \frac{n(n+1)}{2} \frac{(\bar{g}+N-1)!}{\bar{g}!(N-1)!}.$$

The number V_G (resp. V_Z) of scalar variables in the slack variables G_k (resp. Z_k) is given by

$$V_G = (n^2 - nq + q^2)\frac{(\bar{g}+N-1)!}{\bar{g}!(N-1)!} \text{ and } V_Z = (mq)\frac{(\bar{g}+N-1)!}{\bar{g}!(N-1)!}$$

for the gain-scheduled cases and by

$$V_G = q^2 + (n^2 - nq)\frac{(\bar{g}+N-1)!}{\bar{g}!(N-1)!} \text{ and } V_Z = mq$$

for the robust cases. For all control cases, the number R_T of LMI rows in the LMIs (10) is given by

$$R_T = n\frac{(f+\bar{g}+N-1)!}{(f+\bar{g})!(N-1)!},$$

and the number R_S of LMI rows in the LMIs (23) is given by

$$R_S = (2n+p+r)\frac{(f+\bar{g}+d+N)!}{(f+\bar{g}+d+1)!(N-1)!}\frac{(f+\bar{g}+N-1)!}{(f+\bar{g})!(N-1)!}.$$

Table 1 shows the total number of scalar variables $V = V_P + V_G + V_Z$ and the total number of LMI rows $R = R_T + R_S$, for the 16 different control designs. Increasing the degree of the HPPDL function increases both the number of variables V and the number of LMI rows R, as can be seen by comparing the rows for $i = 1$ and $i = 2$ in Table 1. On the other hand, using higher degrees of Pólya's relaxations, for the same degree of HPPDL function, yields the same number of variables, but quickly increases the number of LMI rows, as can be seen by comparing the rows for $i = 1$ and $i = 3$.

	Robust				Gain-Scheduled			
	$q=1$		$q=2$		$q=1$		$q=2$	
i	V	R	V	R	V	R	V	R
1	26	54	24	54	28	54	30	54
2	50	172	42	172	56	172	60	172
3	26	375	24	375	28	375	30	375
4	62	402	51	402	70	402	75	402

Table 1. Number of scalar decision variables V and number of LMI rows R.

6. CONCLUSIONS

This paper shows how gain-scheduled and robust $\mathscr{H}_\infty$ static output feedback controllers can be designed for discrete-time polytopic linear time-varying systems with a prescribed bound on the rate of parameter variation. Properties of homogeneous polynomially parameter-dependent Lyapunov matrices and Pólya's relaxations are exploited to construct a family of LMI relaxations that decrease the conservatism of the control synthesis. It is worth to emphasize that several results from the literature can be recovered as special cases of the proposed results.

The main drawback of the proposed static output feedback control synthesis procedure is its current restriction to the case where the measurement is unaffected by the time-varying parameter. Alleviation of this restriction, as well as dynamic output feedback control synthesis, is the topic of future research.

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