Optimal Active Suspension Design via Convex Analysis

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Abstract

In this note the design of active suspension control is approached in a Youla parametrization setting, leading to dynamic output feedback controllers calculated by convex optimization. The controllers so designed have the following advantages over the usual ones: (i) they do not require the availability of the full states vector; (ii) the optimization problem may be cast in a true multiobjective fashion; (iii) the method reveals the limits of performance of the physical system under the given costs and constraints. An example is provided in a two-degree-of-freedom quarter-car model with $H_{\infty}$ optimization criterion.

Keywords: Active Suspension, Multiobjective Control, Convex Optimization, Youla’s Parametrization.

1. Introduction

In the last decades, active suspension design has attracted a lot of attention from control engineers and researchers. The results have been gratifying, because the trade-off among conflicting factors can be more easily managed with actuators that modify dynamic characteristics on line. In [1], it is shown that the most adequate factors for design are ride comfort, road-holding ability and suspension stroke. These criteria have become now the ones most frequently applied in controller design.

The first steps in this subject were given with applications of traditional optimal control techniques, as the LQR (Linear Quadratic Regulator) which allows finding a static state feedback controller that minimizes a quadratic functional. This approach was deeply analyzed in [1, 2, 3, 4].

In practice, however, this technique is not directly applicable, since it requires full state availability for feedback. A state estimation scheme, via Kalman filter, is therefore employed. The closed-loop retains, in this way, the optimality conditions in terms of the $H_2$ norm of the

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transfer matrix from the disturbances to the controlled outputs.

Some drawbacks of this classical approach are:

- All physical constraints and different objectives must be cast in terms of quadratic costs with associated weights. There is no means of directly establishing hard constraints, nor of defining different cost functionals for the different objectives, in such a way that an optimality (or limit of performance) is attained.
- Even with all objectives embedded into a single cost functional, this functional must be quadratic in order to keep the closed-loop optimality in the case of Kalman filter states estimation. Other functionals will lead to sub-optimal solutions, since the separation principle will no longer hold.

A more natural way of formulating the design objectives would be, perhaps, defining:

- The road-holding ability being weighted with an $\mathcal{H}_\infty$ functional (a worst-case measure), since it is a safety specification.
- The suspension stroke being defined by hard constraints, since it is associated with the geometric format of the car.
- The ride comfort being defined by a quadratic functional, considering that the user discomfort may be adequately captured by a “mean” measure.

There is a strong possibility that, given a controller designed within the classical LQ approach, there exist another controller which, while keeping the same road-holding ability measured by an $\mathcal{H}_\infty$ functional and the same suspension stroke, leads to a better ride comfort measured by a quadratic functional, for instance. This means: with that classical controller, the system will probably not attain its limits of performance.

This note presents the application of a controller design methodology based on the Youla parametrization of closed-loop systems in terms of an affine combination with one free parameter [5]. The methodology features:

- The resulting controller has the structure of a linear dynamic output feedback compensator. There is no need of state estimation.
- Several different cost functionals and constraints have convex structure under such parametrization and so may be combined in a single convex multi-objective optimization problem. These include, for instance, $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms, peak gains, step response overshoots, etc.
- In view of such convexity, there are optimization algorithms which theoretically guarantee the achievement of global optima. The resulting controllers, therefore, attain the limits of system performance (the Pareto optima), in the sense that there is no controller which enhances any objective without degrading at least one other objective.

This paper is organized as follows: in section 2, the vehicle model with active suspension is posed. In section 3 a brief sketch of the Youla’s parametrization theory is presented, with its application to the vehicle model. Section 4 presents the controller optimization problem formulation, with a small example of an $\mathcal{H}_\infty$ optimal controller computation.

2. Vehicle Model

The model employed here is represented in figure 1. This kind of model, despite its simplicity, permits the examination of the vehicular suspension relevant characteristics. It would be possible to consider a damping and a spring in parallel with the actuator system, but it makes no great difference with regards to the theoretical applications of this control technique.

The described system consists of one actuator, performing a control force $u$ between the sprung mass (given by $M_{sp}$) and the unsprung mass ($M_{us}$). The tire is assumed to accomplish a force, only in the vertical, proportional to $K_{us}$ (the damping in the frequency range of interest can be set equal to zero, as explained in [6]). All displacements are taken in relation to the static equilibrium point and the excitation is represented by vector $w$.

![Figure 1: Quarter-car model.](image-url)
in which the state $x$ represents $x = [y_{us} y_{sp} \dot{y}_{us} \dot{y}_{sp}]'$ and the matrices $A, B_1$ and $B_2$ are:

$$
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{K_{us}}{M_{us}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
B_1 = \begin{bmatrix}
0 \\
0 \\
\frac{K_{us}}{M_{us}} \\
M_{us}
\end{bmatrix},
B_2 = \begin{bmatrix}
0 & 0 & -\frac{1}{M_{us}} & \frac{1}{M_{sp}} \\
0 & 0 & 1 & 0
\end{bmatrix}'.

The system output $y$ (the variables chosen as measurable) are the displacements of both bodies $M_{us}$ and $M_{sp}$:

$$
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}.
$$

Thus,

$$
z = P_{zw} w + P_{zu} u
$$

$$
y = P_{yu} w + P_{yu} u, \tag{6}
$$

where $P_{zw}, P_{yu}, P_{zu}$ and $P_{yu}$ are transfer matrices from $w$ to $z$ and $y$, and from $u$ to $z$ and $y$ respectively. The controller $K(s)$ with $u = Ky$ makes the closed loop transfer matrix from $w$ to $z$

$$
z = H w, \tag{7}
$$

in which

$$
H = (P_{zw} + P_{zu} K (I - P_{yu} K)^{-1} P_{yu}).
$$

Now, the suspension model must be put in the standard form. The control system proposed is outlined in figure 3, where the output $z$ is already established. Its choice is described in the next section.

3. Youla’s Parametrization of Active Suspension

In this approach, it is convenient to represent the plant model in a more general way (see [7]), known as generalized plant (figure 2), where $w$ represents the exogenous input vector, $u$ the actuator signal vector, $z$ the regulated outputs vector and $y$ the sensor signal vector.

**Figure 2:** The standard block diagram.

This representation explicitly accounts for the disturbance and noise signals coming into the system. Vector $z$ must contain all signals of interest. Thus, specifications can be formulated in terms of $w$ and $z$ only (for an explanation on this topic, see [5]).

The standard plant transfer function matrix $P(s)$ can be partitioned as (the Laplace variable $s$ is omitted for simplicity):

$$
P = \begin{bmatrix}
P_{zw} & P_{zu} \\
P_{yu} & P_{yu}
\end{bmatrix}.
$$

Thus,

$$
z = P_{zw} w + P_{zu} u
$$

$$
y = P_{yu} w + P_{yu} u, \tag{6}
$$

where $P_{zw}, P_{yu}, P_{zu}$ and $P_{yu}$ are transfer matrices from $w$ to $z$ and $y$, and from $u$ to $z$ and $y$ respectively. The controller $K(s)$ with $u = Ky$ makes the closed loop transfer matrix from $w$ to $z$

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**Figure 3:** Standard form.

Indeed, this model is a 2 DOF (degree-of-freedom) system with the output weighted up (see figure 4) [5].

**Figure 4:** System with 2 DOF.

The plant $P_o$ is the transfer matrix from $w, u$ to $y$ obtained from equation (1). It can be partitioned as:

$$
P_o = [P_0^w P_0^u] = \begin{bmatrix}
P_{o1}^w & P_{o1}^u \\
P_{o2}^w & P_{o2}^u
\end{bmatrix}.
$$

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From figure 3 and (8), the partition defined in (5) becomes:

\[ P_{sw} = \begin{bmatrix} W_1 (I - P_o^w) \\
W_2 (P_o^w - P_o^z) \end{bmatrix} ; \quad P_{yw} = \begin{bmatrix} I \\
0 \end{bmatrix} ; \]

\[ P_{zw} = \begin{bmatrix} -W_1 P_o^w \\
W_2 (P_o^w - P_o^z) \\
W_3 I \end{bmatrix} ; \quad P_{yu} = \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix} . \]

For a rational proper matrix \( P \), there is a doubly-coprime factorization ([7, 8]):

\[ P = NM^{-1} = \tilde{M}^{-1} \tilde{N}, \]

\[ \begin{bmatrix} \tilde{X} & -\tilde{Y} \\
-\tilde{N} & M \end{bmatrix} \begin{bmatrix} M & Y \\
N & X \end{bmatrix} = I. \] (9)

Suppose that \( P_{yu} \) has a factorization as in (9) and \( K \) has a coprime factorization \( K = UV^{-1} = \tilde{V}^{-1} \tilde{U} \), so:

**Theorem 1** \( K \) stabilizes \( P \) \( \iff \) \( K \) stabilizes \( P_{yu} \).

**Theorem 2** The set of all of \( K \in RH_\infty \) which stabilize \( P_{yu} \) is parametrized by:

\[ \begin{aligned}
K &= (Y - MQ)(X - NQ)^{-1} \\
&= (\tilde{X} - Q \tilde{N})^{-1}(\tilde{Y} - Q \tilde{M}) \\
&\in RH_\infty .
\end{aligned} \] (10)

The solution to the controller synthesis problem is established in terms of the following stable transfer matrices:

\[ \begin{aligned}
T_1 &= P_{sw} + P_{zu} M \tilde{Y} P_{yw} , \\
T_2 &= P_{zu} M , \\
T_3 &= M P_{yu} .
\end{aligned} \] (11)

**Theorem 3** The matrices \( T_1, T_2 \) and \( T_3 \) belong to \( RH_\infty \). With \( K \) given by (10), the transfer matrix from \( w \) to \( z \) is

\[ H = T_1 - T_2 QT_3 . \] (12)

The affine relation (12) is employed in the search for the optimal closed-loop system (the so-called model-matching problem). This problem is convex in parameter \( Q \). Once found the optimal \( Q \), the optimal \( K \) is found with (10).

The proofs of these theorems can be found in [7] and [8].

### 4. Controller Optimization

In this section, the controller design methodology will be presented along with the development of a simple example of optimization of an \( H_\infty \) cost functional.

The active system must be able to follow low frequency signal (a tracking problem), to keep relative short suspension stroke and to offer passenger comfort. An adequate choice of the regulated output vector \( z = [r - v_2, v_2 - v_1, u] \); for the exogenous input \( w = [r] \); for the measured output \( y = [r, v_2, v_1] \); and for the controlled input \( u \) is the control force \( u = [u] \). The \( v \) vector components are \( v_1 = y_{us} \) and \( v_2 = y_{sp} \).

It is known (see [7]) that the minimization of the worst case of a quadratic functional cost like

\[ J = (11 - \alpha v_1^2 + \beta v_2^2 + \gamma u^2) \]

with signal \( r \) belonging to the class of signals \( \{ r : r = W w, w \in \mathcal{H}_2, \|w\|_2 \leq 1 \} \) is equivalent to minimize \( \|H\|_\infty \) where \( z = Hw; w \) and \( z \) defined as above.

The optimization problem can be cast in a convex form

\[ \min_{K \in RH_\infty} \phi(H) \equiv \min_{Q \in \Omega} \psi^*(Q) \]

in which the functionals \( \phi \) and \( \psi \) are:

\[ \begin{aligned}
\Omega^* &= \{ Q \in RH_\infty : \psi^*(Q) \leq \alpha \} \\
\phi^*(Q) &= \phi(T_1 - T_2 QT_3) \\
\psi^*(Q) &= \psi(T_1 - T_2 QT_3)
\end{aligned} \]

where \( \phi(\cdot) \) and \( \psi(\cdot) \) can be any convex functional.

This model-matching problem can be solved by the Ritz N-dimensional approximation (for an exposition of the method and algorithm, see [5])

\[ \mathcal{H}_N = \left\{ H : H = T_1 - \sum_{i=1}^{N} T_2 X_i Q_i T_3, \quad X_i \in \mathcal{R}^3, \; Q_i \in RH_\infty \right\} \]

with

\[ Q_i = \left( a \over s + a \right) x_i, \quad \begin{bmatrix} x_{i1} \; x_{i2} \; x_{i3} \end{bmatrix} . \]

The particular choice of \( T_1, T_2 \) and \( T_3 \), as well as the parameter \( a \), the number of base elements in the Ritz approximation and the numerical procedure used to solve the optimization problem will affect the solution obtained (unless the optimum is reached, which would require infinite dimension controllers). Since the aim here is only

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to give a qualitative idea of the design via convex analysis, the details concerning the numerical solution of this particular example will be omitted.

In the example, the asymptotic tracking error is imposed as a constraint on the sprung mass displacement and the control force is limited. The $H_\infty$ objective function is then minimized. Of course, any additional convex constraint could be also incorporated.

The weighting values are: $W_1 = 1$, $W_2 = 1$ and $W_3 = 8 \times 10^{-6}$, i.e., static weightings, but they could be frequency dependent. For example, an analysis of the passive system (see figure 5) yields an appropriate choice of the weighting functions.

$$\text{In this sense}, \ W_1, \ \text{the weighting function on the tracking error} \ r - v_2, \ \text{could be chosen to select the frequency range} \ 0 - 1 \text{ Hz.}$$

The design data adopted in the simulation (also used in [2]) are for the active system: $M_{us} = 28.58 \ [\text{Kg]}$; $M_{sp} = 288.9 \ [\text{Kg]}$; $K_{us} = 155900 \ [\text{N/m}]$, for the passive system: $K_{sp} = 19960 \ [\text{N/m}]$; $C_{sp} = 1861 \ [\text{Ns/m}]$, the data used for the Ritz approximation are: $a = 15$ and $N = 5$ (a small dimension has been chosen in order to allow fast simulation). The coprime factorization has been chosen in such a way that the eigenvalues of $T_1$ are: $-49.68 \pm 102.93j$, $-49.68 \pm 102.93j$ and $-3.98 \pm 11.41j$.

The results of simulation, in which LQR stands for the results in [2], are:

As it can be seen, the proper choice of weightings, constraints and objective functions will determine the dynamic behavior of the closed loop system. In this sense, the methodology presented provides an alternative design technique for active suspension systems.
5. Conclusion

This note has proposed the application of the paradigm of dynamic output feedback controller design via convex multiobjective optimization of the affine “Youla parameter” to active suspension control.

It has been shown that under this paradigm the physical model constraints and objectives may be more naturally expressed, so leading to closed-loops which attain the system limits of performance.

The system is easily implementable, only needing the measurements of the absolute mass displacements of both bodies.

The proposed methodology may be also understood as a “benchmark” for the other control design methods. Even in the case of another controller structure being preferred (perhaps due to the necessity of implementing low order or static controllers), the designer should also perform a design following the guidelines sketched here, in order to get knowledge about the system limits of performance (a by-product of the present methodology). In this way, one could have an absolute measure in order to evaluate all other design methods.

References


Figure 9: Time response to step input.