Comparison between SOS and (S)DSOS Lyapunov functions for nonlinear systems

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Overview

Context

1. Generally challenging problems
2. Polynomial nonnegativity is a NP hard problem
3. Semidefinite programming (SDP)
4. Linear matrix inequality (LMI)
5. Sum-of-squares decomposition
6. Diagonally dominant matrix
7. Region-of-Attraction (ROA) estimation
8. [Tan-Packard, 2004]

Polynomial definitions

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Monomial

Consider $m(x) = x_1^{\theta_1} x_2^{\theta_2} \ldots x_n^{\theta_n}$, $\theta \in \mathbb{Z}_+^n$, then $\deg(m) := \sum_{i=1}^{n} \theta_i, \theta_i \in \mathbb{Z}_+$.

Polynomial

$p(x) \in \mathcal{R}_n$ then $p(x) = \sum_{j=1}^{k} c_j m_j(x)$.

Sum of Squares (SOS)

$s(x) \in \Sigma_n$ then $s(x) = \sum_{i=1}^{k} p_i(x)^2$, and $s(x) \geq 0 \; \forall \; x \in \mathbb{R}^n$.

Polynomial definitions

Multiplicative Monoid - $\mathcal{M}(\cdot)$

The set generated by the all finite products of $g_j \in \mathcal{R}_n$ taken from the set $\{g_1, \cdots, g_l\}$, including the empty product, is called the *Multiplicative Monoid* $\mathcal{M}(g_1, \cdots, g_l)$

Cone - $\mathcal{P}(\cdot)$

We name $\mathcal{P}(p_1, \cdots, p_k)$ as the *Cone* generated by $(p_1, \cdots, p_k)$ if
$\mathcal{P}(p_1, \cdots, p_k) := \left\{ s_0 + \sum_{i=1}^{t} s_i b_i \mid t \in \mathbb{Z}_+, s_i \in \Sigma_n,\ b_i \in \mathcal{M}(p_1, \cdots, p_k) \right\}$.

Ideal - $\mathcal{I}(\cdot)$

We name *Ideal* the set given by:
$\mathcal{I}(h_1, \cdots, h_m) := \left\{ \sum_{j=1}^{m} h_j f_j \mid f_j \in \mathcal{R}_n,\ h_j \in \mathcal{R}_n \right\}$

Positivstellensatz

Theorem (Positivstellensatz)

Given polynomials $\{p_1, \cdots, p_k\}$, $\{g_1, \cdots, g_l\}$ *and* $\{h_1, \cdots, h_m\}$ *in* $\mathcal{R}_n$, *the following statements are equivalent:*

1. $\left\{x \in \mathbb{R}^n \mid p_1(x) \geq 0, \cdots, p_k(x) \geq 0, g_1(x) \neq 0, \cdots, g_l(x) \neq 0, h_1(x) = 0, \cdots, h_m(x) = 0\right\}$ *is empty*
2. *There exist polynomials* $p \in \mathcal{P}(p_1, \cdots, p_k)$, $g \in \mathcal{M}(g_1, \cdots, g_l)$ *and* $h \in \mathcal{I}(h_1, \cdots, h_m)$ *such that:*

$$p + g^2 + h = 0$$

Diagonal dominance and Scaled diagonal dominance

If $s(x) \in \Sigma_n$, there exists $Q = Q' \geq 0 \in \mathbb{R}^{n \times n}$ such that:

$$s(x) = v(x)' Q v(x)$$

where $v(x)$ is the vector of all monomials of $s(x)$.
A sufficient condition is to require that Q is a diagonal dominant matrix or a scaled diagonal dominant [Majumdar, 2014].

Diagonal dominance and Scaled diagonal dominance

Diagonal dominance (DSOS)

$Q \in \mathbb{R}^{n \times n}$ is diagonally dominant ($Q \in DSOS_n$) if $q_{ii} \geq \sum_{i \neq j} |q_{ij}| \; \forall \; i$

Scaled diagonal dominance (SDSOS)

$Q \in \mathbb{R}^{n \times n}$ is scaled diagonally dominant matrix ($Q \in SDSOS_n$) if exist an element-wise positive vector $\gamma \in \mathbb{R}^n$ such that $q_{ii} \geq \sum_{i \neq j} |q_{ij}| \gamma_j \; \forall \; i$

The fact that $Q \in DSOS_n$ or $Q \in SDSOS_n$ implies that $Q \geq 0$ follows directly from Gershgorin's circle theorem.

Search for Lyapunov function via SOS programming

Consider a continuous-time system with state-space equations given by:

$$\dot{x} = f(x)$$

The existence of a Lyapunov function $V(x) > 0$ such that $\nabla V(x) \cdot f(x) < 0,\ \forall x \in (\mathbb{R}^n - \{0\})$, implies that the system is asymptotically globally stable.
Otherwise, there exist points such that $\nabla V \cdot f(x) \geq 0$. In these cases, we shall search for a compact set that excludes such points.

Search for Lyapunov function via SOS

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Consider Φ_α such that:

$$\Phi_\alpha = \{x \in \mathbb{R} \mid V(x) \leq \alpha,\ \nabla V \cdot f(x) < 0\}$$

So Φ_α is a region of attraction for the system. Now, suppose a set $P_\beta \subseteq \Phi_\alpha$ such that:

$$P_\beta := \{x \in \mathbb{R}^n \mid p(x) \leq \beta\}$$

By maximizing the parameter β, P_β is enlarged and become a better approximation of Φ_α.

Now we can state that:

$$\{x \in \mathbb{R}^n \mid V(x) \leq \alpha, \nabla V \cdot f(x) \geq 0, x \neq 0\} \text{ is empty (from } \Phi_\alpha)$$
$$\{x \in \mathbb{R} \mid p(x) \leq \beta, V(x) \geq \alpha, V(x) \neq \alpha\} \text{ is empty (from } P_\beta)$$

by rearranging:

$$\{x \in \mathbb{R}^n \mid \alpha - V(x) \geq 0, \nabla V \cdot f(x) \geq 0, x \neq 0\} \text{ is empty}$$
$$\{x \in \mathbb{R} \mid \beta - p(x) \geq, V(x) - \alpha \geq 0, V(x) \neq \alpha\} \text{ is empty}$$

Search for Lyapunov function via SOS

Applying the Positivstellensatz theorem we obtain the following optimization problem:

Optimization problem for ROA estimation in continuous-time

$$\max \beta \text{ over } s_1, s_2, s_8 \in \Sigma_n,\ V \in \mathcal{R}_n$$
$$V - l_1 \in \Sigma_n$$
$$-\left\{(\beta - p(x))s_8 + (V - \alpha)\right\} \in \Sigma_n$$
$$-\left\{s_1(\alpha - V) + s_2\big(\nabla V \cdot f(x)\big) + l_2\right\} \in \Sigma_n$$

An important part of this algorithm is the choice of degrees for the variables s_1, s_2 and s_8 to match the problem dimensions.
More details in [Tan-Packard, 2004].

Search for Lyapunov function via SOS

In discrete-time case $\nabla V \cdot f(x)$ is replaced by $V \circ f\left(x_k\right) - V$.

Optimization problem for ROA estimation in discrete-time

$\max \beta$ over $s_1, s_2, s_8 \in \Sigma_n,\ V \in \mathcal{R}_n$ such that

$$V - l_1 \in \Sigma_n$$

$$-\left\{\left(\beta - p\left(x_k\right)\right)s_8 + (V - \alpha)\right\} \in \Sigma_n$$

$$-\left\{s_1(\alpha - V) + s_2\left(V \circ f\left(x_k\right) - V\right) + l_2\right\} \in \Sigma_n$$

Reverse time Van der Pol Oscillator example

The mathematical model from Van der Pol Oscillator is given by the second-order differential equation: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ From now on, we will consider $\mu = 1$.

Continuous time

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = (x_1^2 - 1)x_2 + x_1 \end{cases}$$

Reverse time Van der Pol Oscillator example

Discrete time

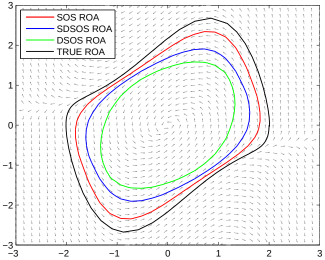
$x_{k+1} = F_1 x_k + F_3 x_k^{[3]}$
$x_k = [x_{1k} \;\; x_{2k}]'$
$x_k^{[i]}$ express the *i-th* Kronecker power [Bacha, 2006].

Choosing the sampling period $T = 0.05s$:

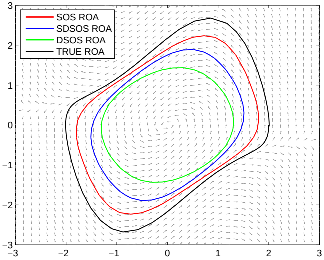
$$F_1 = \begin{bmatrix} 0.9988 & -0.0488 \\ 0.0488 & 0.9500 \end{bmatrix}, \qquad F_3 = \begin{bmatrix} 0 & -0.0012 & 0_{2\times 6} \\ 0 & 0.0488 & \end{bmatrix}$$

Reverse time Van der Pol Oscillator example

ROA estimation with Lyapunov $4^{th} degree$



Continuous time



Discrete time

Reverse time Van der Pol Oscillator example

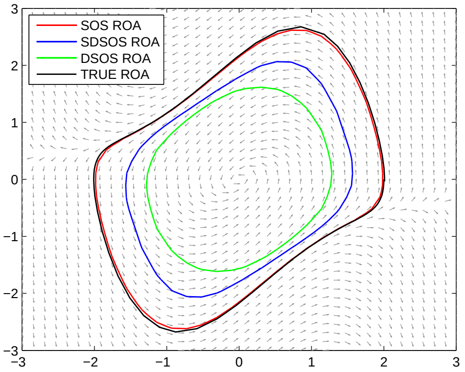
Continuous time performance

Degree	Time (seconds) SOS	DSOS	SDSOS
2	42.34	55.14	44.26
4	957.21	233.00	269.30
6	1570.20	253.50	309.76

Discrete time performance

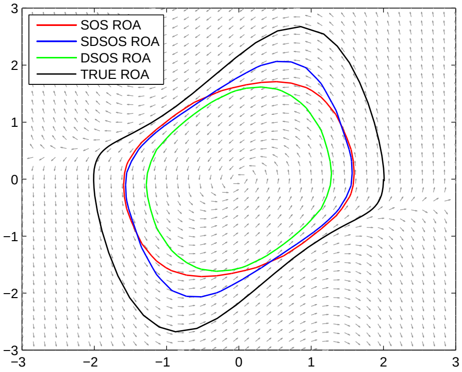
Degree	Time (seconds) SOS	DSOS	SDSOS
4	3074.21	222.22	227.66
8	23989.49	519.51	1073.60

Continuous-time Lyapunov function of 6^{th} degree after convergence



Reverse time Van der Pol Oscillator example

Continuous-time Lyapunov function of 6^{th} and same computational time



Questions?

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