FREQUENCY DOMAIN ANALYSIS OF DYNAMIC SYSTEMS

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Contents

Laplace transform

- The Laplace transform of the function $f(t) : \mathbb{R} \to \mathbb{C}$ denoted as $\hat{f}(s)$ or $\mathcal{L}(f)$ is a function of complex variable

$$\hat{f}(s) : \mathcal{D}(\hat{f}) \to \mathbb{C}$$

where $\mathcal{D}(\hat{f})$ is its domain and

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt \quad (1)$$

$$\mathcal{D}(\hat{f}) := \{s \in \mathbb{C} \ : \ \hat{f}(s) \text{ exists } \} \quad (2)$$

  - It is important to keep in mind that $\hat{f}(s)$ exists means that the integral in (1) converges and is finite.

Laplace transform

- Generally $\mathcal{D}(\hat{f})$ is a strict subset of $\mathbb{C}$. In this case, there exists $s \in \mathbb{C}$ such that $s \notin \mathcal{D}(\hat{f})$ and hence, the determination of the domain $\mathcal{D}(\hat{f})$ is an essential issue when dealing with Laplace transform.
  - Important : The domain of the Laplace transform $\mathcal{D}(\hat{f})$ strongly depends on the domain of the function $f(t)$. As it will be clear in the sequel :

$$\begin{aligned} t \in [0, +\infty) &\implies \mathrm{Re}(s) \in (\alpha, \infty) \\ t \in (-\infty, 0] &\implies \mathrm{Re}(s) \in (-\infty, \beta) \\ t \in (-\infty, \infty) &\implies \mathrm{Re}(s) \in (\alpha, \beta) \end{aligned}$$

for some $\alpha, \beta \in \mathbb{R}$.

Laplace transform

- For each function the Laplace transform (if any) is given :
  - $f(t) = e^{-at} : \mathbb{R} \to \mathbb{C}$ and $\mathcal{D}(\hat{f}) = \emptyset$.
  - $f(t) = e^{-at} : [0, +\infty) \to \mathbb{C}$ and

$$\hat{f}(s) = \frac{1}{s+a}\ , \quad \mathcal{D}(\hat{f}) = \{s \in \mathbb{C}\ :\ \mathrm{Re}(s) > -\mathrm{Re}(a)\}$$

  - $f(t) = e^{-at} : (-\infty, 0] \to \mathbb{C}$ and

$$\hat{f}(s) = -\frac{1}{s+a}\ , \quad \mathcal{D}(\hat{f}) = \{s \in \mathbb{C}\ :\ \mathrm{Re}(s) < -\mathrm{Re}(a)\}$$

  - $f(t) = e^{-a|t|} : (-\infty, +\infty) \to \mathbb{C}$ and

$$\hat{f}(s) = -\frac{2a}{s^2 - a^2}\ , \quad \mathcal{D}(\hat{f}) = \{s \in \mathbb{C}\ :\ |\mathrm{Re}(s)| < \mathrm{Re}(a)\}$$

Definition and domain determination

- The exponential function $e^{-\lambda t} : \mathbb{R} \to \mathbb{C}$ for any $\lambda \in \mathbb{C}$ does not admit a Laplace transform. Hence, for functions with domain $t \in \mathbb{R}$ the Laplace transform is too restrictive, being useless for solving linear differential equations. To circumvent this difficulty, let us restrict our interest to functions defined for $t \in [0, +\infty)$, in which case we have

$$\hat{f}(s) := \int_0^\infty f(t) e^{-st} dt$$

with domain of the general form

$$\mathcal{D}(\hat{f}) := \{s \in \mathbb{C} \ : \ \mathrm{Re}(s) > \alpha\}$$

for some $\alpha \in \mathbb{R}$ to be adequately determined.

Definition and domain determination

- Important class : There exists $s_f \in \mathbb{C}$ such that the limit

$$\lim_{T\to\infty} \int_0^T |f(t)e^{-s_f t}| dt$$

exists and is finite.

Lemma (Domain characterization)

For the functions of this class the following hold :

- *Any $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) \geq \mathrm{Re}(s_f)$ belongs to $\mathcal{D}(\hat{f})$.*
- *There exists M finite such that $|\hat{f}(s)| \leq M$ for all $s \in \mathcal{D}(\hat{f})$.*

Definition and domain determination

- General form : Functions defined for all $t \geq 0$:

$$\mathcal{D}(\hat{f}) := \{s \in \mathbb{C} \ : \ \mathrm{Re}(s) > \alpha\}$$

  - Domain determination : Given a function $f(t)$, determine the minimum value of $\alpha \in \mathbb{R}$ such that

$$\lim_{T \to \infty} \int_0^T |f(t)e^{-\alpha t}| dt < \infty$$

  - Domain determination : Given a function $\hat{f}(s)$, determine the minimum value of $\alpha \in \mathbb{R}$ such that $\hat{f}(s)$ remains analytic in all points of the complex plane belonging to $\mathcal{D}(\hat{f})$.

Definition and domain determination

- The function $\hat{f}(s) = \frac{e^{-s}}{s}$ is not analytic at $s = 0$. Its Laurent series is

$$\hat{f}(s) = \frac{1}{s} - 1 + \frac{s}{2} - \frac{s^2}{6} + \cdots$$

consequently

$$\mathcal{D}(\hat{f}) := \{s \in \mathbb{C} \ : \ \mathrm{Re}(s) > 0\}$$

- The function $\hat{f}(s) = \frac{1-e^{-s}}{s}$ is analytic at $s = 0$. Its Taylor series is

$$\hat{f}(s) = 1 - \frac{s}{2} + \frac{s^2}{6} - \cdots$$

consequently

$$\mathcal{D}(\hat{f}) := \{s \in \mathbb{C} \ : \ \mathrm{Re}(s) > -\infty\} = \mathbb{C}$$

Definition and domain determination

- Rational function :

$$\hat{f}(s) := \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i}$$

where $m \leq n$, $b_i \in \mathbb{R}$ for all $i = 1, \cdots, m$ and $a_i \in \mathbb{R}$ for all $i = 1, \cdots, n$. If $m = n$ it is called proper otherwise strictly proper. It is not analytic at the poles $p_i, i = 1, \cdots, n$ roots of $D(s) = 0$. Hence

$$\alpha = \max_{i=1,\cdots,n} \mathrm{Re}(p_i)$$

- Unitary (Dirac) impulse :

$$\hat{\delta}(s) = 1 \ , \quad \mathcal{D}(\hat{\delta}) = \mathbb{C}$$

Definition and domain determination

- Several calculations involving Laplace transform depend on the precise determination of its domain :
  - Integral : The integral of a function $f(t)$ defined for all $t \geq 0$ can be determined from

$$\int_0^{\infty} f(t)dt = \hat{f}(0)$$

whenever $0 \in \mathcal{D}(\hat{f})$.

  - Limit : The limit of a function $f(t)$ defined for all $t \geq 0$ can be determined from

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\hat{f}(s)$$

whenever $0 \in \mathcal{D}(s\hat{f})$.

Properties

- Basic properties for dynamic systems analysis, valid for functions defined in the time domain $t \geq 0$ and scalars $\theta_1, \theta_2, \cdots$.
  - Linearity :

$$\mathcal{L}\left(\sum_i \theta_i f_i(t)\right) = \sum_i \theta_i \hat{f}_i(s)$$

  - Continuous time convolution :

$$\mathcal{L}(f(t) * g(t)) = \hat{f}(s)\hat{g}(s)$$

  - Time derivative :

$$\mathcal{L}(\dot{f}(t)) = s\hat{f}(s) - f(0)$$

Properties

- Since the functions we are dealing with are only defined for all $t \geq 0$, the time derivative property must be better qualified at $t = 0$.
    - Time derivative : Defining the function

$$h(t) := \begin{cases} \dot{f}(t) & , t > 0 \\ \text{finite value} & , t = 0 \end{cases}$$

generally $h(0) = \lim_{t \to 0^+} \dot{f}(t) = \dot{f}(0^+) < \infty$.

Lemma (Time derivative)

The Laplace transform of $h(t)$ defined above is such that :

$$\hat{h}(s) = s\hat{f}(s) - f(0) \ , \ \ \mathcal{D}(\hat{h}) = \mathcal{D}(s\hat{f})$$

Properties

- Unfortunately, the previous result does not take into account the possibility that $f(t)$ varies arbitrarily fast at $t = 0$. That is, $f(t)$ is not continuous at $t = 0$, which implies that $f(0) \neq 0$. Let us consider this situation using the sequence of functions :

$$f_n(t) := f(t) - f(0)\left(1 + \frac{t}{\tau_n}\right) e^{-t/\tau_n}\ , \quad \forall\ t \geq 0$$

  where $\tau_n > 0$ and goes to zero as n goes to infinity.
  - $f_n(0) = 0$ for all $n \in \mathbb{N}$.
  - $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t > 0$, consequently

$$\lim_{n\to\infty} \hat{f}_n(s) = \hat{f}(s),\ \forall s \in \mathcal{D}(\hat{f})$$

Properties

- Denoting the time derivative of $f(t)$ and of $f_n(t)$ with respect to $t > 0$ as $h(t)$ and $h_n(t)$ respectively, from the previous Lemma we obtain $\hat{h}_n(s) = s\hat{f}_n(s) - f_n(0)$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} \lim_{n\to\infty} \hat{h}_n(s) &= s\hat{f}(s) \\ &= (s\hat{f}(s) - f(0)) + f(0) \\ &= \hat{h}(s) + f(0) \end{aligned}$$

yielding

$$\lim_{n\to\infty} h_n(t) = h(t) + f(0)\delta(t)$$

  - The quantity $\lim_{n\to\infty} h_n(t)$ is called generalized derivative of $f(t)$. It coincides with the time derivative for $\forall\ t > 0$ and is different at $t = 0$ whenever $f(0) \neq 0$.

Properties

- The Laplace transform of the generalized derivative is obtained by multiplying its Laplace transform by s. Let us make clear this concept using the step function defined as $\upsilon(t) = 1$ for all $t \geq 0$

$$\hat{\upsilon}(s) = \frac{1}{s}\ , \quad \mathcal{D}(\hat{\upsilon}) = \{s \in \mathbb{C}\ ;\ \mathrm{Re}(s) > 0\}$$

  - Time derivative : $\hat{h}(s) = s\hat{\upsilon}(s) - 1 = 0$ in accordance to the fact that $h(0) = 0$ and $h(t) = \dot{\upsilon}(t) = 0$ for all $t > 0$.
  - Generalized derivative : $\lim_{n\to\infty} \hat{h}_n(s) = s\hat{\upsilon}(s) = 1$ in accordance to the fact that $\lim_{n\to\infty} h_n(t) = \delta(t)$ for all $t \geq 0$.

Time invariant systems

- Consider a time invariant system defined by the following input-output model

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i}(t) = \sum_{i=0}^{m} b_i \frac{d^i g}{dt^i}(t)$$

with given initial conditions $\frac{d^i y}{dt^i}(0)$, for all $i = 0, \cdots, n-1$. It is assumed that all coefficients are real, $n \leq m$ and that $a_n \neq 0$. The Laplace transform, taking into account the impulse effect on the right hand side, yields

$$\hat{y}(s) = \underbrace{H_0(s)}_{initial\ \ conditions} + H(s)\hat{g}(s)$$

Time invariant systems

- The main facts are as follows :
  - $h_0(t) := \mathcal{L}^{-1}(H_0(s))$ is the part of the solution depending exclusively on the initial conditions.
  - $h(t) := \mathcal{L}^{-1}(H(s))$ is the impulse response (under zero initial conditions). The function $h(t) * g(t)$ is the part of the solution depending exclusively on the input.

$$\Downarrow$$

$$y(t) = h_0(t) + \int_0^t h(t-\tau)g(\tau)d\tau \ , \quad \forall \ t \geq 0$$

- From the state space realization (A, B, C, D) we get

$$H_0(s) := C(sI - A)^{-1}x_0 \ , \quad H(s) := C(sI - A)^{-1}B + D$$

Time varying systems

- We consider the class of time varying systems characterized by

$$\sum_{i=0}^{n} a_i(t)\frac{d^i y}{dt^i}(t) = 0 \ , \quad \forall\ t \geq 0$$

where :

  - The time varying coefficients are such that $a_i(t) = \alpha_i t + \beta_i$ with $\alpha_i, \beta_i \in \mathbb{R}$ for all $i = 1, \cdots, n$ and $\alpha_n \neq 0$.
  - The initial conditions $\frac{d^i y}{dt^i}(0)$, $i = 0, \cdots, n-1$ are not all zero.

- The Laplace transform reveals that whenever $s \in \mathcal{D}(\hat{f})$ it is true that

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}\hat{f}(s)$$

Time varying systems

- Hence, taking into account that

$$\mathcal{L}\left\{\sum_{i=0}^{n}\alpha_i t\frac{d^i y}{dt^i}(t)\right\} = -\frac{d}{ds}\mathcal{L}\left\{\sum_{i=0}^{n}\alpha_i\frac{d^i y}{dt^i}(t)\right\}$$

and not considering for the moment the initial conditions, the Laplace transform provides

$$Q(s)\hat{y}(s) - P(s)\frac{d}{ds}\hat{y}(s) = 0$$

where

$$P(s) := \sum_{i=0}^{n}\alpha_i s^i \;,\;\; Q(s) := \sum_{i=0}^{n}\beta_i s^i - \sum_{i=1}^{n} i\alpha_i s^{i-1}$$

Time varying systems

- Assuming that the roots $p_1, \cdots, p_n$ of $P(s) = 0$ are distinct, partial decomposition yields

$$\frac{Q(s)}{P(s)} = d_0 + \sum_{j=1}^{n} \frac{d_j}{(s - p_j)}$$

where $d_0, \cdots d_n \in \mathbb{C}$. Consequently

$$\frac{1}{\hat{y}(s)} \frac{d}{ds} \hat{y}(s) = d_0 + \sum_{j=1}^{n} \frac{d_j}{(s - p_j)}$$

gives

$$\ln(\hat{y}(s)) = d_0 s + \sum_{j=1}^{n} d_j \ln(s - p_j)$$

Time varying systems

- The Laplace transform of the solution is

$$\hat{y}(s) = e^{d_0 s} \prod_{j=1}^{n} (s - p_j)^{d_j}$$

Important facts :

- If $d_1, \cdots, d_n \in \mathbb{Z}$ with $\sum_{j=1}^{n} d_j \leq 0$ and $d_0 \leq 0$, the above product denoted $H(s)$ is a rational function which provides

$$y(t) = \begin{cases} 0 & 0 \leq t \leq -d_0 \\ h(t + d_0) & t > -d_0 \end{cases}$$

- The above solution $\hat{y}(s)$ may hold even though the initial conditions are not null.

Time varying systems

- Consider the Bessel differential equation

$$t\ddot{y}(t) + \dot{y}(t) + ty(t) = 0 \ , \ y(0) = 1, \ \dot{y}(0) = 0$$

From the same algebraic manipulations we get

$$\frac{1}{\hat{y}(s)} \frac{d}{ds}\hat{y}(s) = \frac{-1/2}{(s+j)} + \frac{-1/2}{(s-j)}$$

$$\Downarrow$$

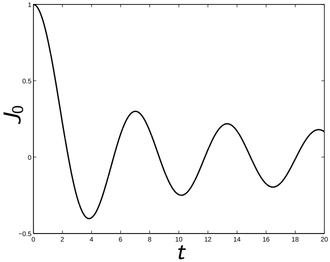
$$\hat{y}(s) = \frac{1}{\sqrt{s^2+1}} \ , \ \mathcal{D}(\hat{y}) = \{s \in \mathbb{C} \ : \ \mathrm{Re}(s) > 0\}$$

and finally $y(t) = J_0(t)$ for all $t \geq 0$ - the Bessel function.

Time varying systems

- Important facts :
  - $J_0(t)$ is determined numerically by series expansion or by solving the Bessel differential equation.
  - The Bessel function has the following convolutional property

$$J_0(t) * J_0(t) = \sin(t) \ , \quad \forall \ t \geq 0$$



Nonrational transforms

- An important function on this matter is the Γ-function, defined for all $r > 0$ by

$$\Gamma(r) := \int_0^\infty \xi^{r-1} e^{-\xi} d\xi$$

Hence $\Gamma(1) = 1$ and

$$\begin{aligned} \Gamma(r+1) &= \xi^r e^{-\xi} \Big|_\infty^0 + r \int_0^\infty \xi^{r-1} e^{-\xi} d\xi \\ &= r\Gamma(r) \end{aligned}$$

shows that for $r \in \mathbb{N}$, $\Gamma(r+1) = r!$. It generalizes the factorial to positive real numbers. A particularly important value is

$$\Gamma(1/2) = \sqrt{\pi}$$

Nonrational transforms

- Considering the function $g(t) := t^r$ defined for all $t > 0$, and $\xi := st$ we have

$$\begin{aligned} \hat{g}(s) &= \int_0^{\infty} t^r e^{-st} dt \\ &= \frac{\Gamma(r+1)}{s^{r+1}} \end{aligned}$$

For all $r > -1 \in \mathbb{R}$ the Laplace transform of $g(t)$ is given by

$$\hat{g}(s) = \frac{\Gamma(r+1)}{s^{r+1}} \ , \quad \mathcal{D}(\hat{g}) = \{s \in \mathbb{C} \ : \ \mathrm{Re}(s) > 0\}$$

This property holds even though $r + 1$ is not an integer number. In this case $\hat{g}(s)$ is not rational.

Nonrational transforms

- Particular cases :
  - For $r = 0$, $g(t) = \upsilon(t)$ is the unit step function and the formula provides
$$\hat{g}(s) = \frac{1}{s}$$
  - For $r = -1/2$, $g(t) = 1/\sqrt{t}$ and the formula provides
$$\hat{g}(s) = \frac{\sqrt{\pi}}{\sqrt{s}}$$
    It can also be concluded that $g(t) = 1/\sqrt{\pi t}$ exhibits the following convolutional property
$$g(t) * g(t) = \upsilon(t) \ , \quad \forall \ t > 0$$

$\mathcal{Z}$ transform

- The $\mathcal{Z}$ transform of the function $f(k) : \mathbb{Z} \to \mathbb{C}$ denoted as $\hat{f}(z)$ or $\mathcal{Z}(f)$ is a function of complex variable

$$\hat{f}(z) : \mathcal{D}(\hat{f}) \to \mathbb{C}$$

where $\mathcal{D}(\hat{f})$ is its domain and

$$\hat{f}(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k} \tag{3}$$

$$\mathcal{D}(\hat{f}) := \{z \in \mathbb{C} \ : \ \hat{f}(z) \text{ exists } \} \tag{4}$$

  - It is important to keep in mind that $\hat{f}(z)$ exists means that the sum in (3) converges and is finite.

$\mathcal{Z}$ transform

- Generally $\mathcal{D}(\hat{f})$ is a strict subset of $\mathbb{C}$. In this case, there exists $z \in \mathbb{C}$ such that $z \notin \mathcal{D}(\hat{f})$ and hence, the determination of the domain $\mathcal{D}(\hat{f})$ is an essential issue when dealing with $\mathcal{Z}$ transform.
  - Important : The domain of the $\mathcal{Z}$ transform $\mathcal{D}(\hat{f})$ strongly depends on the domain of the function $f(k)$. As it will be clear in the sequel :

$$\begin{aligned} k \in [0, +\infty) &\implies |z| \in (\beta, \infty) \\ k \in (-\infty, 0] &\implies |z| \in (0, \alpha) \\ k \in (-\infty, \infty) &\implies |z| \in (\beta, \alpha) \end{aligned}$$

for some positive $\alpha, \beta \in \mathbb{R}$.

$\mathcal{Z}$ transform

- Define the complex sequence $\{z^0, z^1, z^2, \cdots\}$ where $z \in \mathbb{C}$ and notice that

$$\sum_{k=0}^{i-1} z^k = \frac{1 - z^i}{1 - z}, \quad \forall\ i \geq 1$$

Using this we get the following result which is of particular importance on $\mathcal{Z}$ transform calculations :

Lemma (Fundamental lemma)

Consider $z \in \mathbb{C}$. The equality

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}$$

holds and is finite if and only if $|z| < 1$.

$\mathcal{Z}$ transform

- For each function the $\mathcal{Z}$ transform (if any) is given :
  - $f(k) = a^k : \mathbb{Z} \to \mathbb{C}$ and $\mathcal{D}(\hat{f}) = \emptyset$.
  - $f(k) = a^k : [0, +\infty) \to \mathbb{C}$ and

$$\hat{f}(z) = \frac{z}{z-a}\ , \quad \mathcal{D}(\hat{f}) = \{z \in \mathbb{C}\ :\ |z| > |a|\}$$

  - $f(k) = a^k : (-\infty, 0] \to \mathbb{C}$ and

$$\hat{f}(z) = -\frac{a}{z-a}\ , \quad \mathcal{D}(\hat{f}) = \{z \in \mathbb{C}\ :\ |z| < |a|\}$$

  - $f(k) = a^{|k|} : (-\infty, +\infty) \to \mathbb{C}$ and

$$\hat{f}(z) = \frac{(a - 1/a)z}{(z-a)(z-1/a)}\ , \quad \mathcal{D}(\hat{f}) = \{z \in \mathbb{C}\ :\ |a| < |z| < 1/|a|\}$$

Definition and domain determination

- The geometric function $\mu^k : \mathbb{Z} \to \mathbb{C}$ for any $\mu \in \mathbb{C}$ does not admit a $\mathcal{Z}$ transform. Hence, for functions with domain $k \in \mathbb{Z}$ the $\mathcal{Z}$ transform is too restrictive, being useless for solving linear difference equations. To circumvent this difficulty, let us restrict our interest to functions defined for $k \in [0, +\infty)$, in which case we have

$$\hat{f}(z) := \sum_{k=0}^{\infty} f(k) z^{-k}$$

with domain of the general form

$$\mathcal{D}(\hat{f}) := \{z \in \mathbb{C} \ : \ |z| > \beta\}$$

for some positive $\beta \in \mathbb{R}$ to be adequately determined.

Definition and domain determination

- Important class : There exists $z_f \in \mathbb{C}$ such that the limit

$$\lim_{\ell \to \infty} \sum_{k=0}^{\ell} |f(k) z_f^{-k}|$$

exists and is finite.

Lemma (Domain characterization)

For the functions of this class the following hold :

- *Any $z \in \mathbb{C}$ satisfying $|z| \geq |z_f|$ belongs to $\mathcal{D}(\hat{f})$.*
- *There exists M finite such that $|\hat{f}(z)| \leq M$ for all $z \in \mathcal{D}(\hat{f})$.*

Definition and domain determination

- General form : Functions defined for all $k \geq 0 \in \mathbb{Z}$:

$$\mathcal{D}(\hat{f}) := \{z \in \mathbb{C} \ : \ |z| > \beta\}$$

  - Domain determination : Given a function $f(k)$, determine the minimum value of $\beta \in \mathbb{R}$ such that

$$\lim_{\ell \to \infty} \sum_{k=0}^{\ell} |f(k) z_f^{-k}| < \infty$$

  - Domain determination : Given a function $\hat{f}(z)$, determine the minimum value of $\beta \in \mathbb{R}$ such that $\hat{f}(z)$ remains analytic in all points of the complex plane belonging to $\mathcal{D}(\hat{f})$.

Definition and domain determination

- Rational function :

$$\hat{f}(z) := \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{m} b_i z^i}{\sum_{i=0}^{n} a_i z^i}$$

where $m \leq n$, $b_i \in \mathbb{R}$ for all $i = 1, \cdots, m$ and $a_i \in \mathbb{R}$ for all $i = 1, \cdots, n$. If $m = n$ it is called proper otherwise strictly proper. It is not analytic at the poles $p_i, i = 1, \cdots, n$ roots of $D(z) = 0$. Hence

$$\beta = \max_{i=1,\cdots,n} |p_i|$$

- Unitary (Schur) impulse : $\delta(k) := 0^k$, $k \in \mathbb{Z}$

$$\hat{\delta}(z) = 1 \ , \quad \mathcal{D}(\hat{\delta}) = \mathbb{C}$$

Definition and domain determination

- Several calculations involving $\mathcal{Z}$ transform depend on the precise determination of its domain :
  - Sum : The sum of a function $f(k)$ defined for all $k \geq 0$ can be determined from
$$\sum_{k=0}^{\infty} f(k) = \hat{f}(1)$$
whenever $1 \in \mathcal{D}(\hat{f})$.
  - Limit : The limit of a function $f(k)$ defined for all $k \geq 0$ can be determined from
$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1)\hat{f}(z)$$
whenever $1 \in \mathcal{D}((z-1)\hat{f})$.

Properties

- Basic properties for dynamic systems analysis, valid for functions defined in the time domain $k \geq 0$ and scalars $\theta_1, \theta_2, \cdots$.
  - Linearity :
    $$\mathcal{Z}\left(\sum_i \theta_i f_i(k)\right) = \sum_i \theta_i \hat{f}_i(z)$$
  - Discrete time convolution :
    $$\mathcal{Z}(f(k) \bullet g(k)) = \hat{f}(z)\hat{g}(z)$$
  - Step ahead :
    $$\mathcal{Z}(f(k+1)) = z\hat{f}(z) - zf(0)$$

Properties

- Discrete time convolution is essential for dynamic systems analysis, For functions $f(k)$ and $g(k)$ defined for all $k \in [0, +\infty)$ we have

$$
\begin{aligned}
f(k) \bullet g(k) &= \sum_{i=0}^{k} f(k-i)g(i) \\
&= \sum_{i=0}^{k} f(i)g(k-i)\ , \quad \forall\ k \geq 0
\end{aligned}
$$

applying to the discrete impulse function $\delta(k)$ we obtain :

  - $f(k) \bullet \delta(k) = f(k)$ for all $k \geq 0$.
  - Step function : $\upsilon(k) = \sum_{i=0}^{k} \delta(i)$ for all $k \geq 0$.

Time invariant systems

- Consider a time invariant system defined by the following input-output model

$$\sum_{i=0}^{n} a_i y(k+i) = \sum_{i=0}^{m} b_i g(k+i)$$

with given initial conditions $y(i)$, for all $i = 0, \cdots, n-1$. It is assumed that all coefficients are real, $n \leq m$ and that $a_n \neq 0$. The $\mathcal{Z}$ transform yields

$$\hat{y}(z) = \underbrace{H_0(z)}_{\text{initial conditions}} + H(z)\hat{g}(z)$$

Time invariant systems

- The main facts are as follows :
  - $h_0(k) := \mathcal{Z}^{-1}(H_0(z))$ is the part of the solution depending exclusively on the initial conditions.
  - $h(k) := \mathcal{L}^{-1}(H(z))$ is the impulse response (under zero initial conditions). The function $h(k) \bullet g(k)$ is the part of the solution depending exclusively on the input.

$$\Downarrow$$

$$y(k) = h_0(k) + \sum_{i=0}^{k} h(k-i)g(i) \ , \quad \forall \ k \geq 0$$

  - From the state space realization (A, B, C, D) we get

$$H_0(z) := zC(zI - A)^{-1}x_0 \ , \quad H(z) := C(zI - A)^{-1}B + D$$

Time varying systems

- We consider the class of time varying systems characterized by

$$\sum_{i=0}^{n} a_i(k)y(k+i) = 0 \ , \quad \forall\ k \geq 0$$

where :

  - The time varying coefficients are such that $a_i(k) = \alpha_i k + \beta_i$ with $\alpha_i, \beta_i \in \mathbb{R}$ for all $i = 1, \cdots, n$ and $\alpha_n \neq 0$.
  - The initial conditions $y(i)$, $i = 0, \cdots, n-1$ are not all zero.

- The $\mathcal{Z}$ transform reveals that whenever $z \in \mathcal{D}(\hat{f})$ it is true that

$$\mathcal{Z}(kf(k)) = -z\frac{d}{dz}\hat{f}(z)$$

Time varying systems

- Hence, taking into account that

$$\mathcal{Z}\left\{\sum_{i=0}^{n}\alpha_i k y(k+i)\right\} = -z\frac{d}{dz}\mathcal{Z}\left\{\sum_{i=0}^{n}\alpha_i y(k+i)\right\}$$

and not considering for the moment the initial conditions, the $\mathcal{Z}$ transform provides

$$Q(z)\hat{y}(z) - P(z)\frac{d}{dz}\hat{y}(z) = 0$$

where

$$P(z) := \sum_{i=0}^{n}\alpha_i z^{i+1}\ ,\ \ Q(z) := \sum_{i=0}^{n}\beta_i z^i - \sum_{i=1}^{n} i\alpha_i z^i$$

Time varying systems

- Assuming that the roots $p_1, \cdots, p_n$ of $P(z) = 0$ are distinct and noticing that $P(0) = 0$, partial decomposition yields

$$\frac{Q(z)}{P(z)} = \frac{d_0}{z} + \sum_{j=1}^{n} \frac{d_j}{(z - p_j)}$$

where $d_0, \cdots d_n \in \mathbb{C}$. Consequently

$$\frac{1}{\hat{y}(z)} \frac{d}{dz} \hat{y}(z) = \frac{d_0}{z} + \sum_{j=1}^{n} \frac{d_j}{(z - p_j)}$$

gives

$$\ln(\hat{y}(z)) = d_0 \ln(z) + \sum_{j=1}^{n} d_j \ln(z - p_j)$$

Time varying systems

- The $\mathcal{Z}$ transform of the solution is

$$\hat{y}(z) = z^{d_0} \prod_{j=1}^{n} (z - p_j)^{d_j}$$

Important facts :

- If $d_0, d_1, \cdots, d_n \in \mathbb{Z}$ with $\sum_{j=1}^{n} d_j \leq 0$ and $d_0 \leq 0$, the above product denoted $H(z)$ is a rational function which provides

$$y(k) = \begin{cases} 0 & 0 \leq k < -d_0 \\ h(k + d_0) & k \geq -d_0 \end{cases}$$

- The above solution $\hat{y}(z)$ may hold even though the initial conditions are not null.

Time varying systems

- Consider the time varying difference equation

$$(k+1)y(k+1) - (k+1/2)y(k) = 0 \ , \ y(0) = 1$$

From the same algebraic manipulations we get

$$\frac{1}{\hat{y}(z)} \frac{d}{dz}\hat{y}(z) = \frac{1/2}{z} + \frac{-1/2}{(z-1)}$$

$$\Downarrow$$

$$\hat{y}(z) = \sqrt{\frac{z}{z-1}} \ , \ \mathcal{D}(\hat{y}) = \{z \in \mathbb{C} \ : \ |z| > 1\}$$

and finally $y(k) \bullet y(k) = \upsilon(k) \ , \ \forall k \geq 0$. The function $y(k)$ for all $k \geq 0$, can be numerically calculated from the above difference equation.

Problems

1. Consider the Fibonacci difference equation

$$\theta(k+2) - \theta(k+1) - \theta(k) = 0 \ , \quad \theta(0) = 0 \ , \ \theta(1) = 1$$

   - Determine its solution $\theta(k)$ and the output $\theta(k+1) + \theta(k)$.
   - Determine its state space representation.
   - Determine the state space matrices such that the same solution, delayed by one step, is obtained from zero initial condition.

2. For a discrete time linear system with transfer function

$$G(z) = \frac{(z+1)}{(z+1/2)(z-1/2)}$$

   Determine its impulse response.

Problems

3. Consider the second order time varying differential equation

$$\sum_{i=0}^{2}(\alpha_i t + \beta_i)y^{(i)}(t) = 0$$

   - Show that if $\beta_2 = 0$ and $\beta_1 \neq \alpha_2$ then the Laplace transform provides a solution satisfying $y(0) = 0$ and $\dot{y}(0)$ arbitrary.
   - Show that if $\beta_2 = 0$ and $\beta_1 = \alpha_2$ then the Laplace transform provides a solution with $y(0)$ and $\dot{y}(0)$ arbitrary.

4. From the previous problem, determine a second order time varying differential equation and the initial conditions such that the Laplace transform of its solution is

$$\hat{y}(s) = \frac{1}{\sqrt{(s+1)(s+2)}}$$

Problems

5. Consider $z \in \mathbb{C}$. Prove that the equality

$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i$$

holds if and only if $|z| < 1$. Using this result determine the function $f(t)$ defined for all $t \geq 0$ with Laplace transform given by:

- $\hat{f}(s) = \frac{1}{s(1-e^{-s})}$.
- $\hat{f}(s) = \frac{1}{(e^s - e^{-s})}$.
- $\hat{f}(s) = \frac{e^{-s}}{(s+1)(1-e^{-s})}$.

Problems

6. Given $A \in \mathbb{R}^{n \times n}$, determine :
   - $\mathcal{Z}^{-1}\{(zI - A)^{-1}\}$.
   - $\mathcal{Z}^{-1}\{z(zI - A)^{-1}\}$.
   - The $\mathcal{Z}$ transform of $f(k) := \sum_{i=0}^{k} A^i, \forall k \geq 0$.
7. The bilinear transformation is defined by

$$z = \frac{1+s}{1-s}$$

   - Show that the mapping of the region $\mathrm{Re}(s) \leq 0$ in the s-plane is the region $|z| \leq 1$ in the z-plane.
   - Use this property to generalize the Routh criterion to deal with discrete time invariant linear systems.

Problems

8. Consider the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Using the Laplace transform, show that the square matrix

$$\Gamma := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

is such that

$$e^{\Gamma t} = \begin{bmatrix} e^{At} & \int_0^t e^{At} B dt \\ 0 & I \end{bmatrix}$$

9. Define the contour C to be used with the Nyquist criterion for discrete time systems stability analysis.

Problems

10. Consider the time delay system

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) + \kappa y(t - T) = u(t)$$

where $\kappa \geq 0$ and $T = 0, 1, 2$. Using the Nyquist criterion, determine for each T the values of κ preserving asymptotic stability.

11. Consider a time delay system with transfer function

$$H(s) = \frac{1}{s^3 + 4s^2 + 4s + \kappa e^{-Ts}}$$

where $\kappa, T \geq 0$. Determine the stability region (κ, T) using:

- The Nyquist criterion.
- The Routh criterion adopting first and second order approximations to e^{-Ts}.

Problems

12. Consider a time delay system with characteristic equation

$$P(s) + \kappa e^{-Ts} = 0$$

where $\kappa, T \geq 0$. Assuming the roots of $P(s) = 0$ are in the region $\mathrm{Re}(s) < 0$, using the Nyquist criterion show that asymptotic stability is preserved for all $T \geq 0$, whenever

$$\max_{\omega \geq 0} \frac{\kappa}{|P(j\omega)|} < 1$$

Compare to the Nyquist criterion applied with the zero order approximation $e^{-Ts} = 1$.