FREQUENCY DOMAIN ANALYSIS OF DYNAMIC SYSTEMS

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Contents

Partial-fraction decomposition

- Consider a linear continuous-time invariant system with transfer function

$$H(s) = \frac{\hat{z}(s)}{\hat{w}(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i}$$

  with $n \geq m$, $a_i \in \mathbb{R}$, $\forall i = 0, \cdots, n$, $a_n = 1$ and $b_i \in \mathbb{R}$, $\forall i = 0, \cdots, m$. Whenever $n = m$, the function is said to be proper and for $n > m$ it is said to be strictly proper.
- The partial-fraction decomposition consists of determining the scalars α_i such that

$$\frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} = \alpha_0 + \sum_{i=1}^{M} \frac{\alpha_i}{(s - p_i)^{n_i}}$$

  where $\sum_{i=1}^{M} n_i = n$ and p_i are the poles with multiplicity n_i.

Partial-fraction decomposition

- Decompose in partial-fraction the following rational functions

  - $$\frac{s+1}{(s+2)(s+3)(s+5)}$$
  - $$\frac{s+1}{(s+2)^2(s+3)}$$
  - $$\frac{s+1}{(s+2)^3(s+3)}$$
  - $$\frac{s}{(s+1)(s^2+2s+2)}$$

State space realization

- The transfer function $H(s)$ can be written as the following differential equation of n order

$$\sum_{i=0}^{n} a_i \frac{d^i z}{dt^i}(t) = \sum_{i=0}^{m} b_i \frac{d^i w}{dt^i}(t), \ \forall t \geq 0$$

with zero initial conditions. If the initial conditions are not zero, then, applying the Laplace transform on both sides, we obtain

$$\hat{z}(s) = H_0(s) + H(s)\hat{w}(s)$$

where $H_0(s)$ depends only on the initial conditions. Let us denote

$$D[z] = \sum_{i=0}^{n} a_i \frac{d^i z}{dt^i}(t) \ , \ N[z] = \sum_{i=0}^{m} b_i \frac{d^i w}{dt^i}(t)$$

differential operators.

State space realization

Notice that any differential linear equation of order n can be written as n differential equations of first order. This set of equations is named state space realization of the original differential one and has the matrix form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bw(t), \;\; x(0) = x_0 \\ z(t) &= Cx(t) + Dw(t)\end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ and $D \in \mathbb{R}^{1 \times 1}$. Matrices (A, B, C, D) and the initial condition $x_0 \in \mathbb{R}^n$ must be determined such that the function $y(t)$ obtained from the state space realization is identical to the one obtained from the differential equation.

State space realization

- For the equation $D[z] = w$ with $a_n = 1$, defining the state variables

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad x_i(t) = z^{(i-1)}(t), \; i = 1, \cdots, n$$

we obtain

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad D = [0]$$

State space realization

- For the equation $D[z] = N[w]$ with $N[\cdot]$ a differential operator of order $m \leq n-1$, we can rewrite

$$D[\xi] = w \ , \ z = N[\xi]$$

Defining the state variables

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \ x_i(t) = \xi^{(i-1)}(t), \ i = 1, \cdots, n$$

matrices A and B are the same. Moreover,

$$z(t) = E[\xi] = \sum_{i=1}^{m} b_i \xi^{(i)}(t) = \sum_{i=1}^{m} b_i x_{j+1}(t) =$$

allows us to determine

$$C = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & 0 \end{bmatrix}, \ D = [0]$$

State space realization

- Obtain the state space realization of the following systems
  - $H(s) = \dfrac{s^2 + 5s + 3}{s(s^2 + 5s + 6)}$
  - $H(s) = \dfrac{s^2 + 0.1s}{s^2 + 0.1s + 10}$
- Show that for an arbitrary nonsingular matrix $T \in \mathbb{R}^{n \times n}$ the state space realization $(T^{-1}AT, T^{-1}B, CT, D)$ also represents the transfer function $H(s)$ with realization (A, B, C, D).

Bode diagram

- The frequency response of an asymptotically stable system with transfer function $H(s)$ is simply given by $H(j\omega)$.
- It can be represented graphically by diagrams. The most used is the Bode diagram of modulus and phase

$$A(\omega)_{dB} \times \log(\omega), \; \forall \; \omega > 0$$

$$\phi(\omega) \times \log(\omega), \; \forall \; \omega > 0$$

where $A(\omega)_{dB}$ is the modulus of $H(j\omega)$ in decibels and $\phi(\omega)$ is the phase in degrees or radians.

- The modulos of $H(j\omega)$ expressed in decibels is defined as

$$A(\omega)_{dB} = 20\log(|H(j\omega)|)$$

Bode diagram

- Bode diagram can be calculated numerically without any difficulty. However, we can obtain important information about the system by analysing an approximated Bode diagram obtained from the asymptotes. Indeed, notice that writing

$$H(s) = \kappa \frac{\prod_{i=1}^{m}(\gamma_i s + 1)}{\prod_{i=1}^{n}(\tau_i s + 1)}$$

for instance, we obtain

$$A(\omega) = \kappa_{dB} + \sum_{i=1}^{m} |\gamma_i s + 1|_{dB} - \sum_{i=1}^{n} |\tau_i s + 1|_{dB}$$

$$\phi(\omega) = \sum_{i=1}^{m} \angle(\gamma_i s + 1) - \sum_{i=1}^{n} \angle(\tau_i s + 1)$$

for $\gamma_i > 0, \; i = 1, \cdots, m$ and $\tau_i > 0, \; i = 1, \cdots, n$.

Bode diagram

- Considering that $H(s) = \frac{1}{\tau s+1}$, we obtain

$$|H(j\omega)|dB = -20\log\sqrt{(\tau\omega)^2 + 1}$$

where for $\omega >> 1/\tau$ we have

$$|H(j\omega)|_{dB} \approx -20\log(\tau\omega) = -20\log(\omega) - 20\log(\tau)$$

$$\angle H(j\omega) = -90^o$$

and for $\omega << 1/\tau$ we have

$$|H(j\omega)|_{dB} \approx 0, \ \angle H(j\omega) = 0$$

At the frequency $\omega_c = 1/\tau$ occurs the intersection of both asymptotes. This frequency is named cutoff frequency.

Bode diagram

- For second order systems

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

the same idea can be applied, that means, we can obtain the asymptotes by making $\omega << \omega_n$ and $\omega >> \omega_n$. However, generally the approximation for frequencies near to the cuttof one $\omega_c = \omega_n$ may be not good because it depends sensibly on the damping ratio ξ.

Phase and gain margins

- Consider a closed-loop linear system with characteristic equation $1 + C(s)G(s) = 0$ where $G(s)$ represents the system to be controlled and $C(s)$ the controller. Defining $H(s) = C(s)G(s)$ we can calculate, by using the Bode diagram, the following margins:
  - Gain Margin (GM): In the the phase diagram, determine the frequency ω_f such that $\angle H(\omega_f) = -180^o$. By using this frequency, the modulus diagram provides

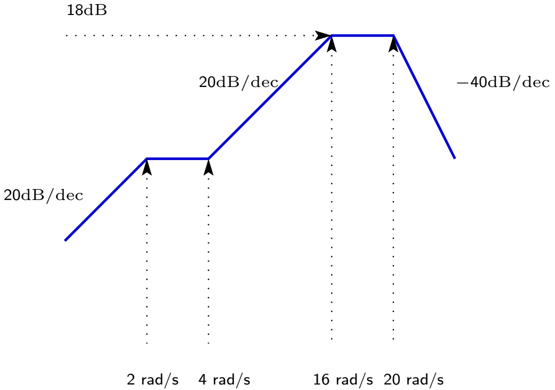
$$GM = -|H(j\omega_f)|_{dB}$$

  - Phase Margin (PM): In the modulus diagram, determine the frequency ω_g such that $|H(\omega_g)|_{dB} = 0\ dB$. By using this frequency, the phase diagram provides

$$PM = 180^o + \angle H(j\omega_g)$$

Bode diagram

- Consider the asymptotic modulus Bode diagram of a minimum phase system presenting one pair of complex poles with $\xi = 0.5$ given by



Bode diagram

- Determine the approximated phase Bode diagram.
- Determine the transfer function $H(s)$.
- Calculate the phase and gain margins (PM and GM).
- Provide the output $z(t)$ in steady state for an input $w(t) = 4\sin(8t)$.

Linear matrix inequalities

- Linear matrix inequalities (LMIs) are essential in the analysis and control design of dynamical systems and to several optimization problems.

Linear Matrix Inequality

An LMI is expressed as

$$\mathcal{A}(x) < 0$$

with

$$\mathcal{A}(x) = A_0 + \sum_{i=1}^{n} A_i x_i$$

where $A_i \in \mathbb{R}^{m \times m}$, $i = 0, \cdots, n$ are symmetric matrices and $x_i \in \mathbb{R}$ is the i-th component of vector x.

Linear matrix inequalities

- Notice that $\mathcal{A}(x) : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a linear function of the vector $x \in \mathbb{R}$.

Convex set

The set of vectors $x \in \mathbb{R}$ satisfying the linear matrix inequality $\mathcal{A}(x) < 0$ is convex.

- Indeed, notice that for two generic points $x_a,\ x_b \in \mathbb{R}^n$ the segment between them is $x = \alpha x_a + (1-\alpha)x_b$ for $0 \leq \alpha \leq 1$. Assuming that $\mathcal{A}(x_a) < 0$ and $\mathcal{A}(x_b) < 0$, we have

$$\begin{aligned} \mathcal{A}(x) &= \mathcal{A}(\alpha x_a + (1-\alpha)x_b) \\ &= \alpha\mathcal{A}(x_a) + (1-\alpha)\mathcal{A}(x_b) \\ &< 0 \end{aligned}$$

where the second equality is due to the fact that $\mathcal{A}(x)$ is linear.

Linear matrix inequalities

- An important result used to linearise some nonlinear constraints is the Schur Complement.

Schur Complement

A linear matrix inequality

$$\mathcal{A}(x) = \begin{bmatrix} S(x) & V(x) \\ V(x)' & Q(x) \end{bmatrix} < 0$$

is equivalent to any of the two nonlinear inequalities

a) $S(x) < 0$ and $Q(x) - V(x)'S(x)^{-1}V(x) < 0$

b) $Q(x) < 0$ and $S(x) - V(x)Q(x)^{-1}V(x)' < 0$

Linear matrix inequalities

- Indeed for part a), notice that $S(x) < 0$ also implies that $S(x)^{-1}$. As a consequence, matrix

$$U(x) = \begin{bmatrix} I & 0 \\ V(x)'S(x)^{-1} & I \end{bmatrix}$$

is nonsingular and allows us to write $\mathcal{A}(x) = U(x)\mathcal{B}(x)U(x)'$, where

$$\mathcal{B}(x) = \begin{bmatrix} S(x) & 0 \\ 0 & Q(x) - V(x)'S(x)^{-1}V(x) \end{bmatrix}$$

Hence matrix $\mathcal{A}(x) < 0$ if and only if $\mathcal{B}(x) < 0$. The proof of part b) is similar.

Linear matrix inequalities

- **Example 1:** Convert the linear inequalities $2x_1 + 3x_2 < 7$, $-x_1 + x_2 < 5$ and $2x_1 - 4x_2 < -4$ in a matrix form.

  Answer:

$$A_0 = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Linear matrix inequalities

- **Example 2:** Convert the nonlinear inequality $(x_1-1)^2+2(x_2-2)^2<5^2$, which is an ellipse with focus in (1,2), in a linear matrix inequality.

  Answer:

  Performing the Schur Complement, we have that it is equivalent to

$$\begin{bmatrix} 2(x_2-2)^2-25 & x_1-1 \\ x_1-1 & -1 \end{bmatrix} < 0$$

  performing it again, we obtain

$$\begin{bmatrix} -25 & x_1-1 & x_2-2 \\ x_1-1 & -1 & 0 \\ x_2-2 & 0 & -1/2 \end{bmatrix} < 0$$

  where matrices $A_0,\ A_1,\ A_2 \in \mathbb{R}^{3\times 3}$ can be directly determined

Linear matrix inequalities

- The concepts we have just presented are important to solve optimization problems described as

$$\inf_{x}\{c'x \ : \ \mathcal{A}(x) < 0\}$$

where $c \in \mathbb{R}^n$.

- In the specific context of control design, two very important problems can be written as the optimization problem just presented.
- The first one is the $\mathcal{H}_2$ norm calculation $\|H\|_2^2$ where $H(s) = C(sI - A)^{-1}B$.

Linear matrix inequalities

- The $\mathcal{H}_2$ norm of $H(s)$ is equal to

$$\|H\|_2^2 = \mathrm{Tr}(B'PB)$$

where $P > 0$ is the solution of the Lyapunov equation $A'P + PA + C'C = 0$. It can calculated through the solution of the optimization problem

$$\|H\|_2^2 = \inf_{X>0} \{\mathrm{Tr}(B'XB) \ : \ A'X + XA + C'C < 0\}$$

or, alternatively, through

$$\|H\|_2^2 = \inf_{Y>0} \{\mathrm{Tr}(CYC') \ : \ AY + YA' + BB' < 0\}$$

Linear matrix inequalities

- The second problem of great importance is the $\mathcal{H}_\infty$ norm.
- It can be shown that the inequality $\|H\|_\infty^2 < \mu$ holds if and only if there exist $P > 0$ and $\mu > 0$ satisfying the Riccati inequality

$$A'P + PA + C'C + (B'P + D'C)'(\mu I - D'D)^{-1}(B'P + D'C) < 0$$

which, applying the Schur Complement can be converted in the following optimization problem

$$\|H\|_\infty^2 = \inf_{P>0,\mu>0} \left\{ \mu \ : \ \begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \mu I \end{bmatrix} < 0 \right\}$$

which is a numerical procedure very efficient for this norm calculation.

Linear matrix inequalities

- Given a system with transfer function

$$H(s) = \frac{s+2}{s^3 + 2.4s^2 + 2.8s + 0.8}$$

  - Obtain the system state space realization.
  - Using the LMILAB from Matlab, solve the optimization problems already provided, in order to calculate $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.
  - Compare the results with the ones obtained by the commands "normh2" and "normhinf" from Matlab.