Continuous-Time Switched Dynamical Systems

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Note to the reader

- This text is based on the following main references :
  - G. S. Deaecto, J. C. Geromel, F. S. Garcia, J. A. Pomílio, "Switched affine systems control design with application to DC-DC Converters", *IET Control Theory & Applications*, vol. 4, pp. 1201-1210, 2010.
  - G. S. Deaecto, G. C. Santos, "State feedback $\mathcal{H}_\infty$ control design of continuous-time switched affine systems", *IET Control Theory & Applications*, vol. 9, pp. 1511-1516, 2015.
  - G. S. Deaecto, "Dynamic output feedback $\mathcal{H}_\infty$ control of continuous-time switched affine systems", *Automatica*, vol 71, pp. 44-49, 2016.
  - G. S. Deaecto, L. N. Egidio, "Practical stability of discrete-time switched affine systems", *Proc. of the European Control Conference*, pp. 2048-2053, 2016.

Switched system

- Consider the continuous-time switched affine system

$$\dot{x}(t) = A_\sigma x(t) + b_\sigma, \;\; x(0) = x_0$$
$$z(t) = E_\sigma x(t)$$

where $b_i, \;\; i \in \mathbb{K}$, is the affine term which allows to the system to have several equilibrium points $x_e \in X_e \subset \mathbb{R}^{n_x}$.

- Defining the auxiliary variable $\xi(t) = x(t) - x_e$ we can rewrite the system as

$$\dot{\xi}(t) = A_\sigma \xi(t) + \ell_\sigma, \;\; \xi(0) = \xi_0$$
$$z_e(t) = E_\sigma \xi(t)$$

where $\ell_\sigma = A_\sigma x_e + b_\sigma$ and $z_e(t) = z(t) - E_\sigma x_e$.

Switched system

Main goal

Our main goal is to design a state dependent switching function $\sigma(x) : \mathbb{R}^{n_x} \to \mathbb{K}$ in order to assure :

- Global asymptotic stability of an equilibrium point x_e inside the set
$$X_e = \{-A_\lambda^{-1} b_\lambda \;:\; A_\lambda \in \mathcal{H}, \lambda \in \Lambda\}$$
where $\mathcal{H}$ is the set of all matrices A_λ Hurwitz stable.
- Assure an upper bound J^{so} for the output $\mathcal{L}_2$ norm
$$\int_0^\infty z_e(t)' z_e(t) dt < J^{so}$$

Stability

Stability

Consider that $x_e \in X_e$ and its associated vector $\lambda \in \Lambda$ are given. If there exist $P > 0$ and symmetric matrices Q_i satisfying

$$A_i' P + P A_i + E_i' E_i + R_i < 0, \ \ i \in \mathbb{K}$$

$$R_\lambda \geq 0$$

then the state dependent min-type switching function

$$\sigma(x) = \arg \min_{i \in \mathbb{K}} (x - x_e)'(-R_i(x - x_e) + 2P\ell_i) \quad (\star)$$

is globally asymptotically stabilizing and satisfies

$$\|z_e\|_2^2 < (x_0 - x_e)' P (x_0 - x_e)$$

Stability

- Indeed, notice that adopting a quadratic Lyapunov function $v(\xi) = \xi' P \xi$ we have

$$\begin{aligned}
\dot{v}(\xi) &= 2\xi' P \dot{\xi} \\
&= 2\xi' P(A_\sigma \xi + \ell_\sigma) + \xi' E_\sigma' E_\sigma \xi - z_e' z_e \\
&< -\xi' R_\sigma \xi + 2\xi' P \ell_\sigma - z_e' z_e \\
&< \min_{i \in \mathbb{K}} \xi'(-R_i \xi + 2P\ell_i) - z_e' z_e \\
&< \min_{\lambda \in \Lambda} \xi'(-R_\lambda \xi + 2P\ell_\lambda) - z_e' z_e \\
&< -z_e' z_e < 0
\end{aligned}$$

  which assures the global asymptotic stability of $\xi = 0$.
- Moreover, integrating both sides from $t = 0$ to $t \to \infty$ and from the fact that $v(\xi(\infty)) = 0$, we obtain

$$\|z_e\|_2^2 < (x_0 - x_e)' P (x_0 - x_e)$$

Stability

- Notice that matrices A_i, $i \in \mathbb{K}$, does not need be Hurwitz as a necessary condition for feasibility.
- For each $x_e \in X_e$ there exists a associated vector $\lambda(x_e) \in \Lambda$

$$\Downarrow$$

Matrix P must be recalculated at each new choice of x_e

- If we impose

$$A_i'P + PA_i + E_i'E_i < 0, \ \ i \in \mathbb{K}$$

the switching rule becomes easier

$$\sigma(x) = \arg\min_{i \in \mathbb{K}} (x - x_e)'P\ell_i \quad (\star\star)$$

and $P > 0$ does not need to be recalculated for each $x_e \in X_e$

$$\Downarrow$$

Matrices A_i must be Hurwitz and admit the same $P > 0$

Examples

In the sequel it will be presented three illustrative examples.

- The first is a buck-boost converter where it was adopted the easier to implement switching rule $(\star\star)$
- The second is a system composed by two stable subsystems where it was adopted the switching rule $(\star)$
- The last one is composed by two unstable subsystems where it was adopted the switching rule $(\star)$

For all of them, we have considered the objective function

$$\inf_{\{P>0, Q_i\} \in \Psi} \mathrm{Tr}(P)$$

where Ψ is the set of all feasible solutions of the associated LMIs (for the first case matrices Q_i, $i \in \mathbb{K}$, do not exist)

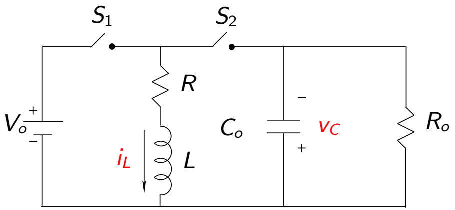
Examples

Example 1 : The next figure presents a Buck-Boost Converter, modeled as $N = 2$ switched affine systems described as

$$A_1 = \begin{bmatrix} -R/L & 0 \\ 0 & -1/R_oC_o \end{bmatrix}, \; A_2 = \begin{bmatrix} -R/L & -1/L \\ 1/C_o & -1/R_oC_o \end{bmatrix}$$

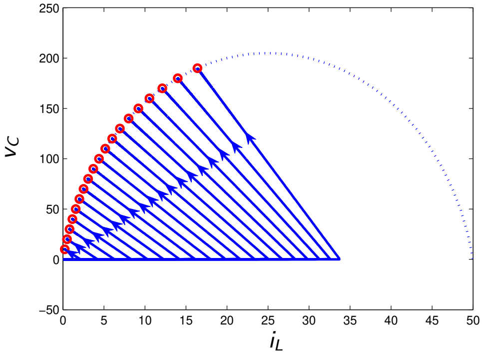
$$b_1 = \begin{bmatrix} V_o/L \\ 0 \end{bmatrix}, \; b_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; E_1 = E_2 = \begin{bmatrix} 0 & 1/\sqrt{R_o} \end{bmatrix}$$

with the state defined as $x = [i_L \; v_C]'$.



Examples

Considering $V_o = 100$ [V], $R = 2$ [Ω], $L = 500\ \mu\mathrm{H}$, $C_o = 470\ \mu\mathrm{F}$ and $R_o = 50\ \Omega$ the next figure shows the phase portrait of the closed-loop system. Trajectories starting from the origin are in contiunous-line. The set X_e is in dotted line.

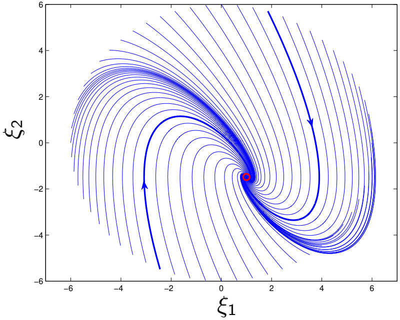


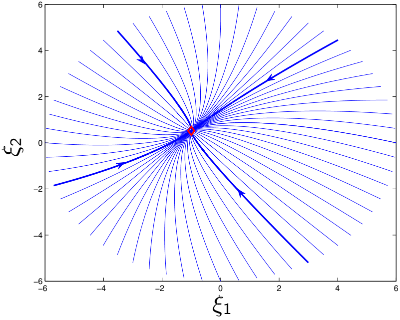
Examples

Example 2 : Consider the switched system defined by matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -2.5 & 0.5 \\ 0.5 & -2.5 \end{bmatrix}, b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

where $E_1 = E_2 = I$ and $\{-1 \pm j\}$, $\{-3, -2\}$ are their eigenvalues, respectively.





Examples

- The chosen equilibrium point was

$$x_e = \begin{bmatrix} 10^{-4} \\ 0.482 \end{bmatrix}, \ \lambda(x_e) = \begin{bmatrix} 0.6506 \\ 0.3494 \end{bmatrix}$$

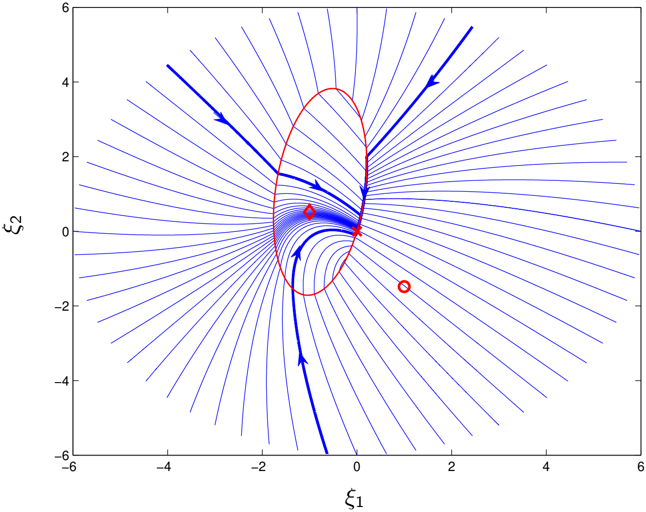
- Solving the conditions for the design of the switching rule $(\star)$ we have obtained $J^{so} = 0.7596$.

- We have obtained the switching surface by finding the geometric place of

$$\xi'(Q_1 - Q_2)\xi + 2\xi' P(\ell_1 - \ell_2) = 0$$

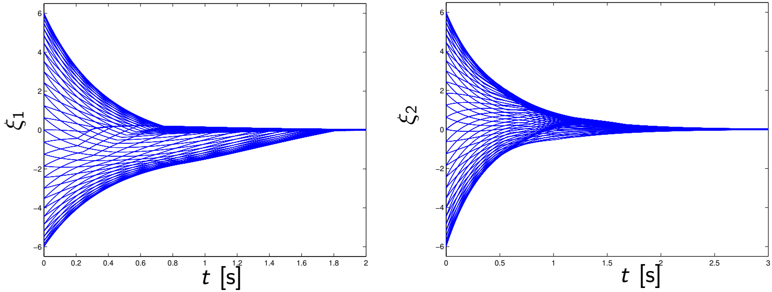
obtaining an **ellipse**.

Phase portrait of the switched system



State trajectories

The state trajectories are presented below :



Solving the conditions related to the switching rule ($\star\star$) we have obtained $J^{so} = 1.6250 >> 0.7596$.

Examples

About this example, the following points are relevant :

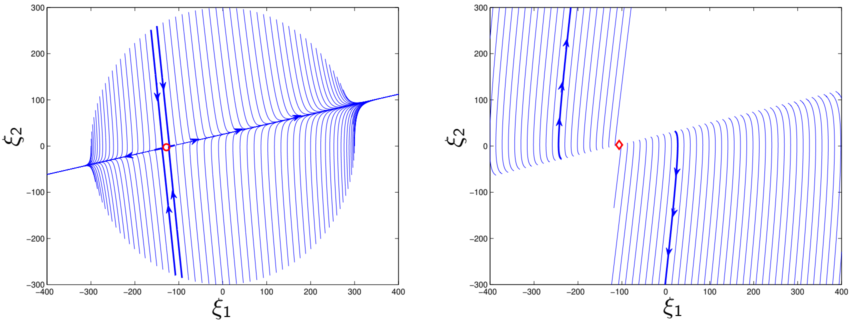
- Different from the linear case, the switched affine system does not present a trivial stabilizing switching rule, since generally the equilibrium point of interest is different from the ones of each isolated subsystem.
- Stable sliding modes generally occur in order to maintain the state trajectories fixed on the desired equilibrium in the steady state.
- If some upper bound is imposed on the switching frequency, asymptotic stability can no be reached. In this case, it is assured practical stability, which occurs when the state trajectories are attracted to a region as small as possible containing the equilibrium.

Examples

Example 3 : Consider the switched system defined by matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 8 \end{bmatrix}, b_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \; b_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, E_1 = E_2 = I$$

and $\{0.217, -9.217\}$, $\{0.258, 7.742\}$ are their eigenvalues.



Examples

- A convex combination is found only for

$$\lambda = \begin{bmatrix} \delta \\ (1-\delta) \end{bmatrix}, \ \delta \in (0.47, \ 0.50)$$

- The chosen equilibrium point was

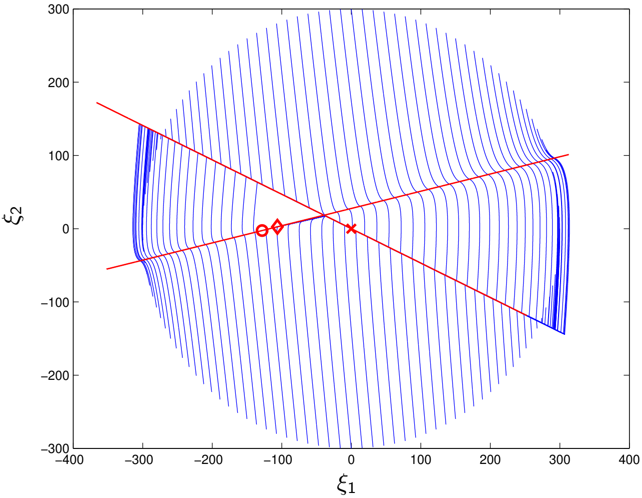
$$x_e = \begin{bmatrix} 105.721 \\ -2.492 \end{bmatrix}, \ \lambda_e = \begin{bmatrix} 0.4984 \\ 0.5016 \end{bmatrix}$$

- Solving the conditions for the design of the rule $(\star)$ we have obtained $J^{so} = 204.298$.
- Finding the geometric place of

$$\xi'(Q_1 - Q_2)\xi + 2\xi' P(\ell_1 - \ell_2) = 0$$

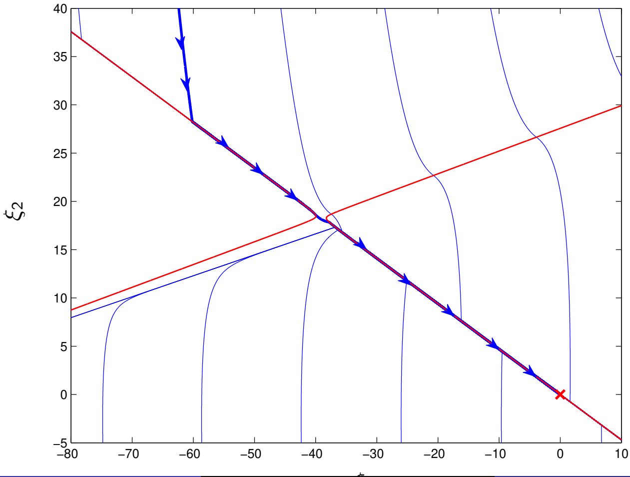
we obtained a **hyperbole** as switching surface.

Phase portrait of the switched system



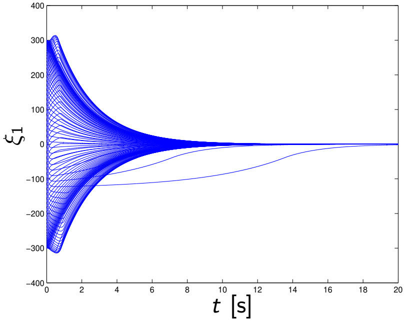
Phase portrait of the switched system

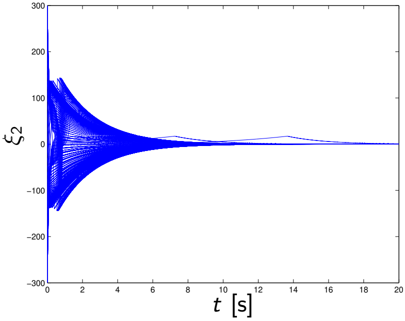
Phase portrait of the switched system with a zoom.



State trajectories

The state trajectories are presented below :





$\mathcal{H}_2$ performance

Let us consider the more general system described as

$$\begin{aligned} \dot{\xi}(t) &= A_\sigma \xi(t) + H_\sigma w(t) + \ell_\sigma, \;\; \xi(0) = 0 \\ z_e(t) &= E_\sigma \xi(t) + G_\sigma w(t) \end{aligned}$$

$\mathcal{H}_2$ performance

For $w(t) = \delta(t)e_k$, the min-type switching function

$$\sigma(x) = \arg\min_{i \in \mathbb{K}} (x - x_e)'(-R_i(x - x_e) + 2P\ell_i)$$

with R_i and P solving the associated LMIs (see pag 6/23) is globally asymptotically stabilizing and satisfies the $\mathcal{H}_2$ cost

$$J_2(\sigma) = \sum_{k=1}^{n_w} \|z_{ek}\|_2^2 < \min_{i \in \mathbb{K}} \mathrm{Tr}(H_i' P H_i)$$

$\mathcal{H}_2$ performance

- In order to obtain this result, it is important to remember that system can be written alternatively as

$$\begin{aligned} \dot{\xi}(t) &= A_\sigma \xi(t) + \ell_\sigma, \ \xi(0) = H_{\sigma(0)} e_k \\ z_e(t) &= E_\sigma \xi(t) \end{aligned}$$

Moreover, from the guaranteed cost of pag 6, we obtain

$$\begin{aligned} J_2(\sigma) &< \sum_{k=1}^{n_w} e_k' H_{\sigma(0)}' P H_{\sigma(0)} e_k \\ &< \min_{i \in \mathbb{K}} \mathrm{Tr}(H_i' P H_i) \end{aligned}$$

where it was supposed that $\sigma(0) = i^*$ is the index that optimizes the right hand side of the inequality.

$\mathcal{H}_\infty$ performance

$\mathcal{H}_\infty$ performance

Consider that $x_e \in X_e$ and its associated vector $\lambda \in \Lambda$ are given. If there exist $P > 0$, symmetric matrices $R_i = R_i'$ and $\rho > 0$ satisfying

$$\begin{bmatrix} A_i'P + PA_i + R_i & \bullet & \bullet \\ H_i'P & -\rho I & \bullet \\ E_i & G_i & -I \end{bmatrix} < 0, \quad R_\lambda \geq 0$$

then the state dependent min-type switching function $(\star)$ is globally asymptotically stabilizing and satisfies

$$J_\infty(\sigma) = \sup_{0 \neq w \in \mathcal{L}_2} \frac{\|z_e\|_2^2}{\|w\|_2^2} < \rho.$$

$\mathcal{H}_\infty$ performance

- Considering once again the Lyapunov function $v(\xi) = \xi' P \xi$ and that the previous conditions are satisfied, we obtain

$$\begin{aligned}
\dot{v}(\xi) &= 2\xi' P (A_\sigma \xi + H_\sigma w + \ell_\sigma) \\
&= \begin{bmatrix} \xi \\ w \end{bmatrix}' \begin{bmatrix} A_\sigma' P + P A_\sigma + E_\sigma' E_\sigma & \bullet \\ H_\sigma' P + G_\sigma' E_\sigma & G_\sigma' G_\sigma - \rho I \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} + 2\xi' P \ell_\sigma \\
&\quad - z_e' z_e + \rho w' w \\
&< -\xi' R_\sigma \xi + 2\xi' P \ell_\sigma - z_e' z_e + \rho w' w \\
&< -z_e' z_e + \rho w' w
\end{aligned}$$

Now integrating both sides from $t = 0$ to $t \to \infty$, and taking into account that $v(\xi(0)) = v(\xi(\infty)) = 0$ we obtain that $J_\infty(\sigma) < \rho$.

$\mathcal{H}_\infty$ performance

- It is important to notice that to obtain the first inequality, it was used the fact that

$$\sup_{w\in\mathcal{L}_2} \begin{bmatrix} \xi \\ w \end{bmatrix}' \underbrace{\begin{bmatrix} A_i'P + PA_i + E_i'E_i & \bullet \\ H_i'P + G_i'E_i & G_i'G_i - \rho I \end{bmatrix}}_{\mathcal{L}_i} \begin{bmatrix} \xi \\ w \end{bmatrix} = \xi \left(A_i'P + PA_i + \right.$$
$$\left. + E_i'E_i + (H_i'P + G_i'E_i)'(\rho I - G_i'G_i)^{-1}(H_i'P + G_i'E_i) \right) \xi$$

  with $\rho I - G_i'G_i > 0$.
- To obtain less conservative conditions we can adopt a state-input dependent switching function

$$\sigma(x,w) = \arg\min_{i\in\mathbb{K}} \begin{bmatrix} \xi \\ w \end{bmatrix}' \mathcal{L}_i \begin{bmatrix} \xi \\ w \end{bmatrix} + 2\xi' P \ell_\sigma$$

  assuring in this case that $\mathcal{L}_\lambda < 0$.

Problems

1) For the Example 3 of pag 17 with

$$H_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ H_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

and the same equilibrium point adopted in the example, determine

a) The $\mathcal{H}_2$ guaranteed cost

b) Matrices P and R_i important to implement the switching function $\sigma(\xi)$.

c) The state trajectories ξ taking into account that $\sigma(0) = i^*$ is the optimum index used to optimize the guaranteed cost.

d) By numerical simulation the actual cost $J_2(\sigma)$.

2) Obtain w^* that maximizes

$$\sup_{w \in \mathcal{L}_2} \begin{bmatrix} \xi \\ w \end{bmatrix}' \mathcal{L}_i \begin{bmatrix} \xi \\ w \end{bmatrix}$$

and show that the equality in pag 26 holds.

Problems

3) For the Example 2 of pag 12 with

$$H = H_i = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}', E_2 = 2E_1, \; G = G_i = 1/2$$

and the same equilibrium point adopted in the example, determine

a) The guaranteed cost ρ considering that the switching rule is state dependent $\sigma(x)$ as well as the associated matrices P, R_i.

b) The guaranteed cost ρ considering that the switching rule is state-input dependent $\sigma(x, w)$ as well as the associated matrix P.

c) The norm $\|E_\lambda(sI - A_\lambda)^{-1}H + G\|_\infty^2$

d) Compare theoretically and discuss the previous results.

e) For the result of item b) obtain the state trajectories taking into account that $w(t) = \mathrm{sen}(2.4t)$ for $t \in [0, 9]$ and $w(t) = 0$ otherwise.

Problems

4) For the more general switched affine system

$$\begin{aligned} \dot{\xi}(t) &= A_\sigma \xi(t) + B_\sigma u(t) + H_\sigma w(t) + \ell_\sigma, \ \xi(0) = 0 \\ z_e(t) &= E_\sigma \xi(t) + F_\sigma u(t) + G_\sigma w(t) \end{aligned}$$

consider that

$$u(t) = K_\sigma \xi$$

Determine :

a) The conditions for the joint design of K_i and $\sigma(x)$ in order to obtain an $\mathcal{H}_2$ guaranteed cost.
b) The conditions for the joint design of K_i and $\sigma(x)$ in order to obtain an $\mathcal{H}_\infty$ guaranteed cost.
c) The conditions for the joint design of K_i and $\sigma(x, w)$ in order to obtain an $\mathcal{H}_\infty$ guaranteed cost.