Continuous-Time Switched Dynamical Systems

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1. CHAPTER II - Switched Linear Systems
   - Min-type Lyapunov function
   - Differentiability
   - Stability
   - Lyapunov-Metzler inequalities
   - Closed-loop performance
   - Consistency
   - State feedback control design
   - Problems

Note to the reader

- This text is based on the following main references :
  - D. Liberzon, *Switching in Systems and Control*, Birkhäuser, 2003.
  - J. C. Geromel and P. Colaneri, "Stability and stabilization of continuous-time switched linear systems", SIAM Journal on Control and Optimization, vol. 45, pp. 1915–1930, 2006.
  - J. C. Geromel and G. S. Deaecto, "Stability analysis of Lur'e-type switched systems", IEEE Transactions on Automatic Control, vol. 59, pp. 3046-3050, 2014.
  - J. C. Geromel, G. S. Deaecto and J. Daafouz, "Suboptimal switching control consistency analysis for switched linear systems", IEEE Transactions on Automatic Control, vol. 58, pp. 1857-1861, 2013.

Switched system

- Consider the switched linear system with state space realization

$$\dot{x} = A_\sigma x, \;\; x(0) = x_0$$
$$z = E_\sigma x$$

where

- $x \in \mathbb{R}^{n_x}$ is the state
- $z \in \mathbb{R}^{n_z}$ is the controlled output and
- $\sigma(\cdot) \; : \; \mathbb{R}^{n_x} \to \{1, \cdots, N\} = \mathbb{K}$ is the switching function to be determined.

Min-type Lyapunov function

- Let us define the min-type Lyapunov function

$$v(x) = \min_{i \in \mathbb{K}} x' P_i x = \min_{\lambda \in \Lambda} \sum_{i \in \mathbb{K}} \lambda_i x' P_i x$$

with matrices $P_i > 0$, $\forall i \in \mathbb{K}$, and the unitary simplex Λ

$$\Lambda = \left\{ \lambda \in \mathbb{R}^N \ : \ \lambda_i \geq 0, \ \sum_{i \in \mathbb{K}} \lambda_i = 1 \right\}$$

- Adopt the following notation for the convex combination of a set of matrices $\{X_1, \cdots, X_N\}$

$$X_\lambda = \sum_{i \in \mathbb{K}} \lambda_i X_i, \ \ \lambda \in \Lambda$$

- Important : Notice that $v(x)$ is positive definite, continuous but not differentiable.

Differentiability

- Danskin theorem is the most important result to deal with derivative of functions described as

$$\phi(x) = \min_{y \in Y} f(x, y)$$

where Y is a compact set and $\nabla_x f(x, y)$ exists.

Danskin theorem

The one-sided directional derivative of $\phi(x)$ exists in any direction d and is given by

$$\begin{aligned} D_+\phi(x, d) &= \lim_{\epsilon \to 0^+} \frac{\phi(x + \epsilon d) - \phi(x)}{\epsilon} \\ &= \min_{y \in Y(x)} \nabla_x f(x, y)' d \end{aligned}$$

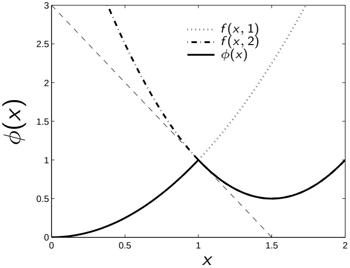
where $Y(x) = \{y \ : \ \phi(x) = f(x, y)\}$.

Differentiability

- **Example :** Consider the function $\phi(x)$ with

$$f(x,1) = x^2, \ f(x,2) = 2(x-3/2)^2 + 1/2$$

defined for all $x \in \mathbb{R}$ and $y \in Y = \{1,2\}$. Using Danskin theorem, determine the one-sided directional derivative of $\phi(x)$ in the direction d in the points $x = 0.5$ and $x = 1$.



Differentiability

- Notice that the function $\phi(x)$ is not differentiable in $x = 1$.
- We have the sets $Y(0.5) = \{1\}$ and $Y(1) = \{1, 2\}$.

Hence, we can calculate

- $D_+\phi(0.5, d) = \min_{y \in Y(0.5)} \nabla f(x, y) d = d$
- $D_+\phi(1, d) = \min_{y \in Y(1)} \nabla f(x, y) d = \min\{2d, -2d\} = -2d$.

Differentiability

- Let us apply Danskin theorem to the min-type function $v(x)$ for an arbitrary trajectory $x(t)$ of the system

$$\dot{x}(t) = A_\sigma x(t)$$

Denote $I(x) = \{i \ : \ v(x) = v_i(x)\}$. We want to calculate

$$\begin{aligned} D_+ v(x(t)) &= \lim_{\epsilon \to 0^+} \frac{v(x(t+\epsilon)) - v(x(t))}{\epsilon} \\ &= \lim_{\epsilon \to 0^+} \frac{v(x(t) + \epsilon A_\sigma x(t))) - v(x(t))}{\epsilon} \\ &= \min_{\ell \in I(x(t))} \nabla_x v_\ell(x(t))' A_\sigma x(t) \\ &= \min_{\ell \in I(x(t))} x(t)'(A_\sigma' P_\ell + P_\ell A_\sigma) x(t) \end{aligned}$$

Differentiability

- Important facts :
  - If we define a switching strategy such that :

$$\sigma(x(t)) = i \in I(x(t))$$

then

$$\begin{aligned} D_+ v(x(t)) &= \min_{\ell \in I(x(t))} x(t)'(A_i' P_\ell + P_\ell A_i)x(t) \\ &\leq x(t)'(A_i' P_i + P_i A_i)x(t) \end{aligned}$$

in which case the upper bound of $D_+ v(x(t))$ is very simple.
  - Whenever the set $I(x(t))$ presents only one element the function $v(x(t))$ is differentiable and the equality holds.
  - For more than one element in $I(x(t))$, sliding modes generally occurs.
  - During the sliding mode, the system presents a particular dynamic which is different from those of the subsystems.

Stability

- Let us study stability by adopting the quadratic Lyapunov function $v(x) = x'Px$ which is the simplest one.

Lemma : Quadratic stability

If there exist a matrix $P > 0$ and a vector $\lambda \in \Lambda$ satisfying

$$A_\lambda' P + PA_\lambda + Q_\lambda < 0$$

with $Q_i = E_i'E_i$ then the min-type switching function

$$\sigma(x) = \arg\min_{i \in \mathbb{K}} x'(A_i'P + PA_i + E_i'E_i)x$$

is globally asymptotically stabilizing and assures that

$$\|z\|_2^2 < x_0'Px_0$$

Stability

- Indeed, notice that the time derivative of $v(x)$ provides

$$\begin{aligned}
\dot{v}(x) &= x'(A'_\sigma P + PA_\sigma + E'_\sigma E_\sigma)x - z'z \\
&= \min_{i\in\mathbb{K}} x'(A'_i P + PA_i + E'_i E_i)x - z'z \\
&= \min_{\lambda\in\Lambda} x'(A'_\lambda P + PA_\lambda + Q_\lambda)x - z'z \\
&\leq x'(A'_\lambda P + PA_\lambda + Q_\lambda)x - z'z \\
&< -z'z
\end{aligned}$$

where the second equality comes from the choice of the switching function and the last inequality is due to the fact that $A'_\lambda P + PA_\lambda + Q_\lambda < 0$.

Stability

- Notice that no stability condition is required from the isolated subsystems $A_i, \ i \in \mathbb{K}$!
- The sufficient condition is the existence of $\lambda \in \Lambda$ such that A_λ is Hurwitz stable. This is a NP hard problem !
- Moreover, integrating the inequality both sides from $t = 0$ to $t \to \infty$, we have

$$\int_0^\infty \dot{v}(x)dt = v(x(\infty)) - v(x(0)) < -\int_0^\infty z(t)'z(t)dt$$

which provides $\|z\|_2^2 < x_0' P x_0$ since the asymptotic stability assures that $v(x(\infty)) = 0$.

Stability

- An important improvement is obtained by adopting the following min-type Lyapunov function

$$v(x) = \min_{i \in \mathbb{K}} x' P_i x$$

and a subclass of Metzler matrices $\Pi = \{\pi_{ji}\} \in \mathcal{M}_c$, $(i,j) \in \mathbb{K} \times \mathbb{K}$, with the following properties

$$\sum_{j \in \mathbb{K}} \pi_{ji} = 0, \quad \pi_{ij} \geq 0, \ \forall j \neq i \in \mathbb{K} \times \mathbb{K}$$

- All matrices belonging to $\mathcal{M}_c$ is such that

$$\pi_{ii} = - \sum_{j \neq i \in \mathbb{K}} \pi_{ji} \leq 0, \ \ i \in \mathbb{K}$$

Stability

- Gershgorin circle theorem : Each eigenvalue of $\Pi \in \mathcal{M}_c$ is inside a circle centered at $(\pi_{ii}, 0)$ and with radius $|\pi_{ii}| = \sum_{j \neq i \in \mathbb{K}} \pi_{ji}$.
- Frobenius-Perron theorem : The null eigenvalue is the one with maximum real part and the associated eigenvector $v \in \mathbb{R}^N$ is nonnegative. Hence the usual normalization $\sum_{i \in \mathbb{K}} v_i = 1$ makes $v \in \Lambda$.
- Notice that for an arbitrary $\nu \in \Lambda$ the matrix

$$\Pi = -I + \nu e'$$

with $e' = [1 \ \ 1 \ \ \cdots \ \ 1]$ is a Metzler matrix of the class $\mathcal{M}_c$.

Stability

Theorem : Stability

If there exist matrices $P_i > 0$ and a Metzler matrix $\Pi \in \mathcal{M}_c$ satisfying the so called Lyapunov-Metzler inequalities

$$A_i' P_i + P_i A_i + \sum_{j \in \mathbb{K}} \pi_{ji} P_j + E_i' E_i < 0$$

Then the min-type switching function

$$\sigma(x) = \arg \min_{i \in \mathbb{K}} x' P_i x$$

is globally asymptotically stabilizing and assures that

$$\|z\|_2^2 < \min_{i \in \mathbb{K}} x_0' P_i x_0$$

Stability

- Defining $I(x) = \{i \ : \ x'P_i x = v(x)\}$, for $i \in I(x)$, and $\Pi \in \mathcal{M}_c$, we have

$$
\begin{aligned}
x'\left(\sum_{j\in\mathbb{K}} \pi_{ji} P_j\right) x &= \pi_{ii} x' P_i x + \sum_{j\neq i} \underbrace{\pi_{ji}}_{\geq 0} x' P_j x \\
&\geq \pi_{ii} x' P_i x + \sum_{j\neq i} \pi_{ji} x' P_i x \\
&\geq \underbrace{\left(\sum_{j\in\mathbb{K}} \pi_{ji}\right)}_{=0} x' P_i x \\
&\geq 0
\end{aligned}
$$

Stability

- Considering that at an arbitrary instant of time $t \geq 0$ we have $\sigma(t) = i \in I(x)$, the one-sided directional derivative of $v(x)$ provides

$$
\begin{aligned}
D_+ v(x) &= \min_{\ell \in I(x)} x'(A_i' P_\ell + P_\ell A_i + E_i' E_i)x - z'z \\
&\leq x'(A_i' P_i + P_i A_i + E_i' E_i)x - z'z \\
&< -x' \underbrace{\left(\sum_{j \in \mathbb{K}} \pi_{ji} P_j \right)}_{\geq 0} x - z'z \\
&< -z'z
\end{aligned}
$$

- Moreover, making the same procedure as before, we have

$$
\|z\|_2^2 < v(x_0) = \min_{i \in \mathbb{K}} x_0' P_i x_0
$$

Stability

At this point, some remarks are in order :

- We can write

$$\left(A_i + \frac{\pi_{ii}}{2} I\right)' P_i + P_i \left(A_i + \frac{\pi_{ii}}{2} I\right) + \sum_{j \neq i \in \mathbb{K}} \pi_{ji} P_j + E_i' E_i < 0$$

$$\Downarrow$$

No stability property is required from the isolated subsystems because $\pi_{ii} \leq 0$.

- The conditions are nonconvex due to the matrices product $\{\pi_{ji}, P_i\}$ and difficult to solve for more than two subsystems.
- The conditions assure stability even in the eventual existence of sliding modes.
- This phenomenon occurs whenever the set $I(x)$ presents more than one element.

Stability

- The classical Filippov's result establishes that whenever the system operates in a sliding mode, it is described by

$$\dot{x} = \sum_{i \in I(x)} \alpha_i A_i x$$

where $\alpha \in \Sigma(x)$ with $\Sigma(x)$ being the set composed by vectors α such that $\alpha_i \geq 0$ and $\sum_{i \in I(x)} \alpha_i = 1$. Hence,

$$\begin{aligned} D_+ v(x) &= \min_{\ell \in I(x)} \sum_{i \in I(x)} \alpha_i x' \left(A_i' P_\ell + P_\ell A_i\right) x \\ &\leq \max_{\alpha \in \Sigma(x)} \min_{\ell \in I(x)} \sum_{i \in I(x)} \alpha_i x' \left(A_i' P_\ell + P_\ell A_i\right) x \\ &\leq \min_{\ell \in I(x)} \max_{\alpha \in \Sigma(x)} \sum_{i \in I(x)} \alpha_i x' \left(A_i' P_\ell + P_\ell A_i\right) x \\ &\leq \underbrace{\max_{i \in I(x)} x' \left(A_i' P_i + P_i A_i\right) x \quad < 0}_{\text{because the previous theorem assures } D_+ v(x) < 0, \ \forall i \in I(x)} \end{aligned}$$

Lyapunov-Metzler inequalities

Modified Lyapunov-Metzler inequalities

The result of the previous theorem remains valid whenever there exist matrices $P_i > 0$ and a positive scalar $\gamma > 0$ satisfying the modified Lyapunov-Metzler inequalities

$$A_i' P_i + P_i A_i + \gamma(P_j - P_i) + E_i' E_i < 0, \ \ i \neq j \in \mathbb{K} \times \mathbb{K}$$

- These conditions were obtained by restricting the Metzler matrices to those with the same main diagonal $\gamma = \sum_{j \neq i} \pi_{ji}$.
- Although they are clearly more conservative, for an arbitrary number of subsystems, they can be solved by LMIs whenever a scalar $\gamma > 0$ is fixed.

Lyapunov-Metzler inequalities

Theorem : Alternative stability conditions

If there exist a matrix $P > 0$, symmetric matrices W_i and a Metzler matrix $\Pi \in \mathcal{M}_c$ satisfying the inequalities

$$A_i' P + P A_i + \sum_{j \in \mathbb{K}} \pi_{ji} W_j + E_i' E_i < 0, \;\; i \in \mathbb{K}$$

Then the min-type switching function

$$\sigma(x) = \arg\min_{i \in \mathbb{K}} x' W_i x$$

is globally asymptotically stabilizing and assures

$$\|z\|_2^2 < x_0' P x_0$$

Moreover $v(x) = x' P x$ is a Lyapunov function for the system.

Lyapunov-Metzler inequalities

- This result is obtained from the Lyapunov-Metzler inequalities with $\Pi(\mu) = \mu\Pi \in \mathcal{M}_c$ and choosing $P_i = P + \mu^{-1}W_i$ with $\mu > 0$, which provide

$$A_i'\underbrace{(P+\mu^{-1}W_i)}_{P_i}+\underbrace{(P+\mu^{-1}W_i)}_{P_i}A_i+\sum_{j\in\mathbb{K}}\mu\pi_{ji}\underbrace{(P+\mu^{-1}W_j)}_{P_j}+E_i'E_i<0$$

- Making $\mu \to \infty$ we have

$$A_i'P + PA_i + \sum_{j\in\mathbb{K}} \pi_{ji}W_j + E_i'E_i < 0, \;\; i \in \mathbb{K}$$

Lyapunov-Metzler inequalities

- The switching function becomes

$$\sigma(x) = \arg\min_{i\in\mathbb{K}} x' \underbrace{P_i}_{P+\mu^{-1}W_i} x = \arg\min_{i\in\mathbb{K}} x' W_i x$$

Notice that the switching rule does not depend directly on the Lyapunov function !

Lyapunov-Metzler inequalities

- The next lemma presents some instrumental results that are very important to obtain stability conditions based on an unique subsystem.

Lemma

Let the symmetric matrices Q_i, $\forall i \in \mathbb{K}$, be given. The following statements are equivalent :

1. There exist matrices $W_i > 0$ and a Metzler matrix $\Pi \in \mathcal{M}_c$ satisfying
$$Q_i + \sum_{j \in \mathbb{K}} \pi_{ji} W_j < 0, \;\; i \in \mathbb{K}$$
2. There exist symmetric matrices R_i and $\nu \in \Lambda$ satisfying $R_\nu = 0$ and
$$Q_i + R_i < 0, \;\; i \in \mathbb{K}$$

Lyapunov-Metzler inequalities

- Indeed, considering that statement 1) is true, choosing

$$R_i = \sum_{j \in \mathbb{K}} \pi_{ji} W_j, \; i \in \mathbb{K}$$

and $\nu \in \Lambda$ as being the eigenvector associated with the null eigenvalue of Π, we have

$$\begin{aligned} R_\nu &= \sum_{i \in \mathbb{K}} \nu_i \sum_{j \in \mathbb{K}} \pi_{ji} W_j \\ &= \sum_{j \in \mathbb{K}} \left(\sum_{i \in \mathbb{K}} \pi_{ji} \nu_i \right) W_j = 0 \end{aligned}$$

and, therefore, statement 2) is true.

Lyapunov-Metzler inequalities

- Now, assuming that statement 2) is true, choosing

$$\Pi = -I + \nu[1 \quad \dots \quad 1] \quad , \quad W_i = W_N + (R_N - R_i)$$

  we have

$$\sum_{j=1}^{N} \pi_{ji} W_j = W_\nu - W_i = -R_\nu + R_i = R_i$$

  because $R_\nu = 0$. Hence, from statement 2) we have that statement 1) is true.

- Using this lemma the alternative stability conditions can be written as follows.

Lyapunov-Metzler inequalities

Corollary : Alternative stability conditions

If there exist a matrix $P > 0$, symmetric matrices R_i and $\nu \in \Lambda$ satisfying $R_\nu = 0$ and the inequalities

$$A_i' P + P A_i + E_i' E_i + R_i < 0, \;\; i \in \mathbb{K}$$

Then the max-type switching function

$$\sigma(x) = \arg \max_{i \in \mathbb{K}} x' R_i x$$

is globally asymptotically stabilizing and assures

$$\|z\|_2^2 < x_0' P x_0$$

Moreover $v(x) = x' P x$ is a Lyapunov function for the system.

Lyapunov-Metzler inequalities

- The inequality follows directly from the previous lemma.
- The switching function is obtained from

$$
\begin{aligned}
\sigma(x) = \arg\min_{i\in\mathbb{K}} x' \underbrace{P_i}_{P+\mu^{-1}W_i} x &= \arg\min_{i\in\mathbb{K}} x' \underbrace{W_i}_{W_N+(R_N-R_i)} x \\
&= \arg\max_{i\in\mathbb{K}} x' R_i x \\
&= \arg\min_{i\in\mathbb{K}} x'(A_i'P + PA_i + E_i'E_i)x
\end{aligned}
$$

- It is simple to see that these conditions are the quadratic stability ones provided in the beginning of this chapter.
- Moreover, they are a particular case of the Lyapunov-Metzler inequalities.

Example 1 - Stability

- Consider a system defined by two unstable subsystems

$$A_1 = \begin{bmatrix} 0 & 1 \\ -5 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 \\ 2 & -5 \end{bmatrix}, \ E_1 = E_2 = I$$

  The equilibrium point of the first subsystem is an unstable focus $\lambda\{A_1\} = \{0.5 \pm 2.1794\}$, while the equilibrium point of the second is a saddle $\lambda\{A_2\} = \{0.3723, \ -5.3723\}$.
- We have solved problem

$$\inf_{P_i > 0, \gamma > 0} \gamma$$

  subject to the Lyapunov Metzler inequalities with

$$P_i - \gamma I < 0, \ i \in \mathbb{K}$$

- Notice that the guaranteed cost is given by

$$\|z\|_2^2 < \min_{i \in \mathbb{K}} x_0' P_i x_0 < \gamma x_0' x_0$$

Example 1 - Stability

- We have obtained $\gamma^* = 1.4482$ and the matrices

$$P_1 = \begin{bmatrix} 1.3428 & 0.2994 \\ 0.2994 & 0.4576 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.3566 & 0.3039 \\ 0.3039 & 0.4401 \end{bmatrix}$$

associated with the choice

$$\Pi = \begin{bmatrix} -p & q \\ p & -q \end{bmatrix}$$

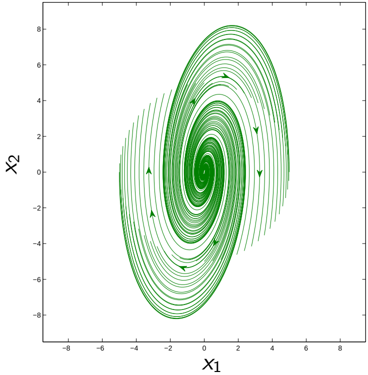
with $(p^*, q^*) = (144, 160)$ determined by unidimensional search inside the box $(p, q) \in [0, 160] \times [0, 160]$ with step 2.

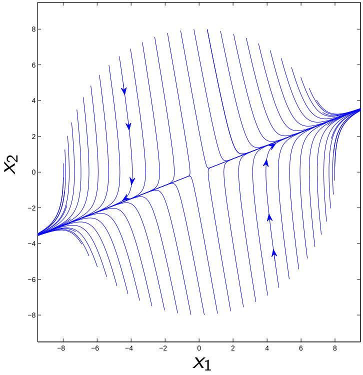
- We have determined the switching surface by making

$$x'(P_1 - P_2)x = 0$$

Example 1 - Stability

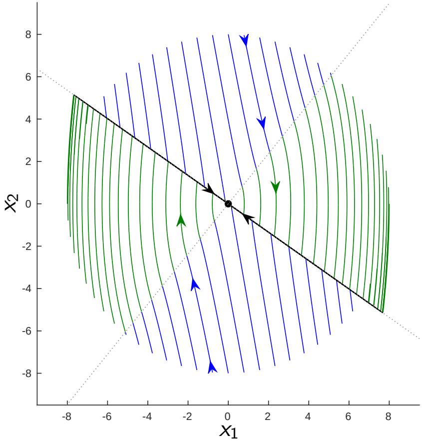
- Phase portrait of both isolated subsystems.





Example 1 - Stability

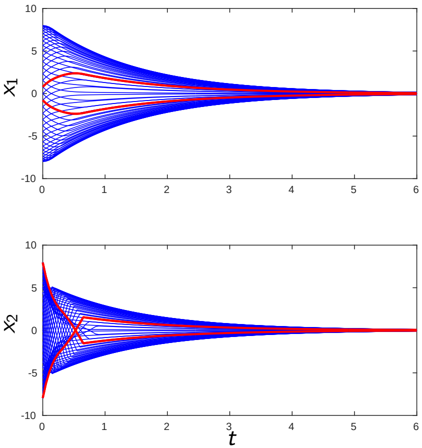
- Phase portrait of the switched system.



It is clear the sliding mode surface and the dynamics of both subsystems !

Example 1 - Stability

- State trajectories of the switched system.



Example 2 - Stability

- Consider a third order switched linear system defined by

$$A_1 = \begin{bmatrix} -3 & -6 & 3 \\ 2 & 2 & -3 \\ \alpha & 0 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 3 & 3 \\ \beta & -3 & -3 \\ 0 & 0 & -2 \end{bmatrix}$$

  and $E_1 = E_2 = I$.

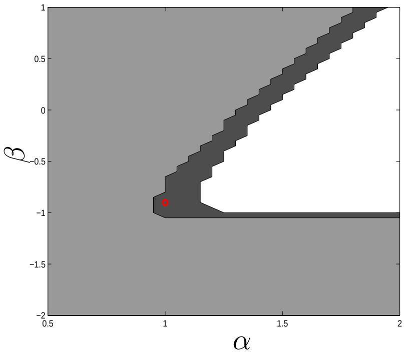
- We have varied the pair α, β inside the interval $[0.5, 2]$, $[-2, 1]$, respectively, analyzing the feasibility of the Lyapunov Metzler inequalities for

$$\Pi = \begin{bmatrix} -p & q \\ p & -q \end{bmatrix}$$

  with (p, q) belonging to the box $[0, 20] \times [0, 20]$.

Example 2 - Lyapunov-Metzler

- The region in gray (dark and light) is the feasibility region for the Lyapunov-Metzler inequalities.
- The region in dark gray does not admit a Hurwitz stable convex combination of the subsystems matrices.



This makes clear that the Lyapunov-Metzler inequalities are less conservative than asking for A_λ be Hurwitz stable!

Example 2 - Lyapunov-Metzler

- For $(\alpha, \beta) = (1.0, -0.9)$ the switched system does not present a stable convex combination of the subsystems matrices. However, matrices

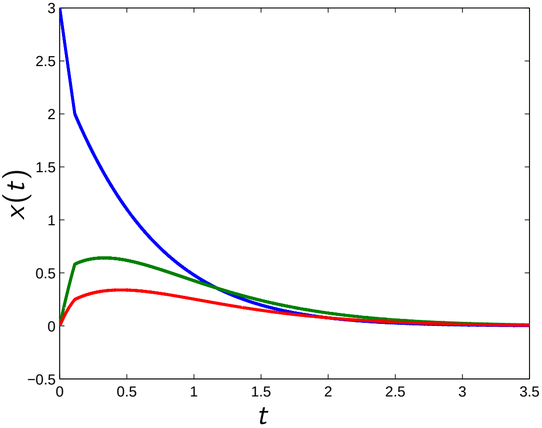
$$P_1 = \begin{bmatrix} 3.6048 & 8.0420 & -6.7034 \\ 8.0420 & 34.4956 & -33.0632 \\ -6.7034 & -33.0632 & 34.3784 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 4.6089 & 4.6781 & -0.4977 \\ 4.6781 & 11.6580 & -12.0200 \\ -0.4977 & -12.0200 & 22.7412 \end{bmatrix}$$

with $(p, q) = (1.86, 1.79)$ satisfy the Lyapunov-Metzler inequalities.

Example 2 - Lyapunov-Metzler

- The state trajectories obtained by implementing the switching rule with matrices P_1, P_2 are presented as follows



Closed-loop performance

Consider now a more general switched linear system described by

$$\mathcal{G}_{\sigma(t)} := \left\{ \begin{array}{rcl} \dot{x}(t) & = & A_{\sigma(t)}x(t) + H_{\sigma(t)}w(t), \ x(0) = 0 \\ z(t) & = & E_{\sigma(t)}x(t) + G_{\sigma(t)}w(t) \end{array} \right.$$

where

- $w(t) \in \mathbb{R}^{n_w}$ is the external input.

In our context we will adopt two classes of external inputs :

- The impulsive type $w(t) = e_k\delta(t)$, for which the dynamic equation can be written alternatively as

$$\dot{x}(t) = A_{\sigma(t)}x(t), \ x(0) = H_{\sigma(0)}e_k$$

  e_k is the k-th column of the identity matrix.
- Those belonging to the set $w \in \mathcal{L}_2$.

Performance indexes

- For a stabilizing given trajectory $\sigma(t)$ we have :
  - $\mathcal{H}_2$ performance index : For $G_i = 0, \ \forall i \in \mathbb{K}$, the controlled output $z(t)$ associated with the external input $w(t) = e_k\delta(t)$, allows us to define the following $\mathcal{H}_2$ index

$$J_2(\sigma) = \sum_{k=1}^{m} \|z_k\|_2^2$$

  - $\mathcal{H}_\infty$ performance index : The controlled output $z(t)$ associated with any arbitrary external input $w(t) \in \mathcal{L}_2$ allows us to define the following $\mathcal{H}_\infty$ index

$$J_\infty(\sigma) = \sup_{0 \neq w \in \mathcal{L}_2} \frac{\|z\|_2^2}{\|w\|_2^2}$$

Both indexes are difficult to be calculated then the idea is to find a suitable upper bound !

Performance indexes

- For a stabilizing given trajectory $\sigma(t)$ we have :
  - $\mathcal{H}_2$ performance index : For $G_i = 0, \ \forall i \in \mathbb{K}$, the controlled output $z(t)$ associated with the external input $w(t) = e_k\delta(t)$, allows us to define the following $\mathcal{H}_2$ index

$$J_2(\sigma) = \sum_{k=1}^{m} \|z_k\|_2^2 = \underbrace{\|E_i(sI - A_i)^{-1}H_i\|_2^2}_{\sigma(t)=i, \forall t \geq 0}$$

  - $\mathcal{H}_\infty$ performance index : The controlled output $z(t)$ associated with any arbitrary external input $w(t) \in \mathcal{L}_2$ allows us to define the following $\mathcal{H}_\infty$ index

$$J_\infty(\sigma) = \sup_{0 \neq w \in \mathcal{L}_2} \frac{\|z\|_2^2}{\|w\|_2^2} = \underbrace{\|E_i(sI - A_i)^{-1}H_i + G_i\|_\infty^2}_{\sigma(t)=i, \forall t \geq 0}$$

Both indexes are difficult to be calculated then the idea is to find a suitable upper bound !

$\mathcal{H}_2$ performance

Theorem : $\mathcal{H}_2$ performance

If there exist matrices P_i, $i \in \mathbb{K}$, and a Metzler matrix $\Pi \in \mathcal{M}_c$ satisfying the Lyapunov-Metzler inequalities

$$A_i'P_i + P_iA_i + \sum_{j\in\mathbb{K}} \pi_{ji}P_j + E_i'E_i < 0$$

then the min-type switching function

$$\sigma(x) = \arg\min_{i\in\mathbb{K}} x'P_ix$$

is globally asymptotically stabilizing and satisfies

$$J_2(\sigma) < \min_{i\in\mathbb{K}} \mathrm{Tr}(H_i'P_iH_i)$$

$\mathcal{H}_2$ performance

- From the previous results we have

$$J_2(\sigma) < \sum_{k=1}^{n_w} \min_{i\in\mathbb{K}} \left(H_{\sigma(0)}e_k\right)' P_i \left(H_{\sigma(0)}e_k\right)$$
$$< \min_{i\in\mathbb{K}} \underbrace{\sum_{k=1}^{n_w} \left(H_{\sigma(0)}e_k\right)' P_i \left(H_{\sigma(0)}e_k\right)}_{\mathrm{Tr}\left(H'_{\sigma(0)}P_iH_{\sigma(0)}\right)}$$
$$< \min_{i\in\mathbb{K}} \mathrm{Tr}\left(H_i' P_i H_i\right)$$

where $\sigma(0) = i$ can be imposed since $\sigma(0)$ is arbitrary.

- The best $\mathcal{H}_2$ guaranteed cost is given by

$$J_2^{so} = \inf_{\{\Pi, P_i\}\in\mathcal{X}_2} \min_{i\in\mathbb{K}} \mathrm{Tr}(H_i' P_i H_i)$$

where $\mathcal{X}_2$ is the set of feasible solutions of the Lyapunov-Metzler inequalities.

$\mathcal{H}_\infty$ performance

Theorem : $\mathcal{H}_\infty$ performance

If there exist matrices P_i, $i \in \mathbb{K}$, a Metzler matrix $\Pi \in \mathcal{M}_c$ and a scalar $\rho > 0$ satisfying the Riccati-Metzler inequalities

$$\begin{bmatrix} A_i'P_i + P_iA_i + \sum_{j\in\mathbb{K}} \pi_{ji}P_j + E_i'E_i & \bullet \\ H_i'P_i + G_i'E_i & -\rho I + G_i'G_i \end{bmatrix} < 0$$

then the min-type switching function

$$\sigma(x) = \arg\min_{i\in\mathbb{K}} x'P_ix$$

is globally asymptotically stabilizing and satisfies

$$J_\infty(\sigma) < \rho$$

$\mathcal{H}_\infty$ performance

- Consider that the Riccati-Metzler inequalities hold. Adopting the min-type Lyapunov function $v(x) = \min_{i\in\mathbb{K}} x'P_ix$ and assuming that $\sigma(t) = i \in I(x(t))$ for a $t \geq 0$, we have

$$\begin{aligned} D_+v(x) &= \min_{\ell\in I(x)} 2(A_ix + H_iw)'P_\ell x \\ &< \begin{bmatrix} x \\ w \end{bmatrix}' \begin{bmatrix} A_i'P_i + P_iA_i & \bullet \\ H_i'P_i & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\ &< -x'\left(\sum_{j\in\mathbb{K}} \pi_{ji}P_j\right)x - z'z + \rho w'w \\ &< -z'z + \rho w'w \end{aligned}$$

where the second inequality comes from the validity of the Riccati-Metzler inequalities.

$\mathcal{H}_\infty$ performance

- Integrating both sides from $t = 0$ to $t \to \infty$ we obtain

$$v(x(\infty)) - v(x(0)) < -\|z\|_2^2 + \rho\|w\|_2^2$$

where the left hand side is null since $v(x(\infty)) = 0$ because the system is stable and $v(x(0)) = 0$ because $x(0) = 0$.

- The best $\mathcal{H}_\infty$ guaranteed cost is given by

$$J_\infty^{so} = \inf_{\{\Pi, P_i, \rho\} \in \mathcal{X}_\infty} \rho$$

where $\mathcal{X}_\infty$ is the set of feasible solutions of the Riccati-Metzler inequalities.

Consistency

- Consistency is an important concept related to stabilizing switching rules. Consider $\alpha = \{2, \infty\}$, define $\mathcal{S}$ as the set of all stabilizing switching rules and $\mathcal{C}$ as the set of all constant rules $\sigma(t) = i \in \mathbb{K}$ for all $t \geq 0$.

Consistency

A switching rule $\sigma_\alpha \in \mathcal{S}$ is said to be consistent whenever it provides a performance better than the one of each isolated subsystem, that is

$$J_\alpha(\sigma_\alpha) \leq J_\alpha(\sigma), \ \sigma \in \mathcal{C}$$

when the inequality is strict the switching rule σ_α is said to be strictly consistent.

Consistency

- As it will be clear in the sequel the min-type switching function is consistent for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ indexes.
- In order to show this, let us notice that
  - Matrix $\Pi = \Pi_0 = 0$ belongs to the subclass of Metzler matrices $\Pi_0 \in \mathcal{M}_c$.
  - Matrix $\Pi = \Theta_\ell$ defined as

$$\pi_{ii} = -\beta, \ \pi_{\ell i} = \beta, \ \forall i \in \mathbb{K}, \ \ell \neq i$$

with $\beta > 0$ also belongs to $\Theta_\ell \in \mathcal{M}_c$. For $N = 4$ and $\ell = 2$:

$$\Pi = \Theta_2 = \begin{bmatrix} -\beta & 0 & 0 & 0 \\ \beta & 0 & \beta & \beta \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{bmatrix}$$

In this case

$$\sum_{j \in \mathbb{K}} \pi_{ji} P_j = \beta (P_\ell - P_i), \ \forall i \in \mathbb{K}$$

Consistency

- For the $\mathcal{H}_2$ performance, since that $\Pi = \Pi_0$ is feasible, we have

$$J_2(\sigma) < \inf_{P_i > 0} \{ \mathrm{Tr}(H_i' P_i H_i) \ : \ A_i' P_i + P_i A_i + E_i' E_i < 0 \}$$
$$< \underbrace{\| E_i (sI - A_i)^{-1} H_i \|_2^2}_{J_2(i)}$$

  which holds for all $i \in \mathbb{K}$.
- Hence, the min-type switching rule is consistent.
- In general, we have $J_2(\sigma) \ll J_2(i)$ which indicates that $\sigma(x)$ is strictly consistent.
- Moreover, with $\Pi = \Pi_0$ we have

$$J_2^{so} = \min_{i \in \mathbb{K}} \| E_i (sI - A_i)^{-1} H_i \|_2^2$$

Consistency

- For the $\mathcal{H}_\infty$ performance, since that $\Pi = \Pi_0$ is feasible, we have

$$
\begin{aligned}
J_\infty(\sigma) &< \inf_{P_i>0,\rho>0} \left\{ \rho : \begin{bmatrix} A_i'P_i + P_iA_i + E_i'E_i & \bullet \\ H_i'P_i + G_i'E_i & -\rho I + G_i'G_i \end{bmatrix} < 0 \right\} \\
&< \inf_{\rho>0} \{ \rho : \underbrace{\|E_i(sI - A_i)^{-1}H_i + G_i\|_\infty^2}_{J_\infty(i)} < \rho \} \\
&< \max_{i\in\mathbb{K}} \|E_i(sI - A_i)^{-1}H_i + G_i\|_\infty^2
\end{aligned}
$$

- Hence, differently from the $\mathcal{H}_2$ case, matrix Π_0 can not be used to prove consistency in the $\mathcal{H}_\infty$ framework.
- Moreover, with $\Pi = \Pi_0$ we have

$$
J_\infty^{so} = \max_{i\in\mathbb{K}} \|E_i(sI - A_i)^{-1}H_i + G_i\|_\infty^2
$$

Consistency

- However, considering $G_i = G,\ i \in \mathbb{K}$ and adopting $\Pi = \Theta_\ell$ with $\ell \in \mathbb{K}$, the Riccati-Metzler inequalities become

$$\begin{bmatrix} A_i' P_i + P_i A_i + E_i' E_i + \beta(P_\ell - P_i) & \bullet \\ H_i' P_i + G' E_i & -\rho I + G' G \end{bmatrix} < 0$$

which is feasible whenever $\beta > 0$ is large enough, $P_i > P_\ell \forall i \neq \ell$ and

$$\begin{bmatrix} A_\ell' P_\ell + P_\ell A_\ell + E_\ell' E_\ell & \bullet \\ H_\ell' P_\ell + G' E_\ell & -\rho I + G' G \end{bmatrix} < 0$$

which is equivalent to

$$\| E_\ell (sI - A_\ell)^{-1} H_\ell + G \|_\infty^2 < \rho$$

Consistency

- Consequently, we can conclude that

$$J_\infty(\sigma) < \inf_{\rho>0}\{\rho : \|E_\ell(sI - A_\ell)^{-1}H_\ell + G\|_\infty^2 < \rho\} < \underbrace{\|E_\ell(sI - A_\ell)^{-1}H_\ell + G\|_\infty^2}_{J_\infty(\ell)}$$

  which holds for all $\ell \in \mathbb{K}$.
- Hence, the min-type switching rule is consistent.
- In general, we have $J_\infty(\sigma) \ll J_\infty(i)$ which indicates that $\sigma(x)$ is strictly consistent.

Example 3 - $\mathcal{H}_2$ performance

- Consider a switched linear system composed of two stable subsystems

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -9 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \ H = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ E_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

we can calculate

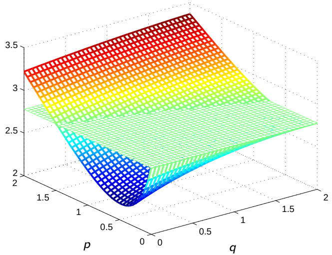
$$\begin{aligned} J_2(\sigma) &= \min_{\ell \in \{1,2\}} \| E_\ell (sI - A_\ell)^{-1} H_\ell \|_2^2 \\ &= \min\{ \underbrace{2.7778}_{\sigma(t)=1, \ \forall t \geq 0}, 25.0000 \} \end{aligned}$$

Example 3 - $\mathcal{H}_2$ performance

- Adopting a Metzler matrix of the form

$$\Pi = \begin{bmatrix} -p & q \\ p & -q \end{bmatrix}$$

we have determined the minimum guaranteed cost for all (p, q) inside the box $[0, 2] \times [0, 2]$ as shown in the next figure where the plane surface concerns $\min_{\sigma \in \mathcal{C}} J_2(\sigma)$.



Example 3 - $\mathcal{H}_2$ performance

- The best guaranteed cost was obtained for

$$\Pi^* \approx \begin{bmatrix} -0.45 & 0 \\ 0.45 & 0 \end{bmatrix} \Rightarrow J_2^{so} = 2.1929$$

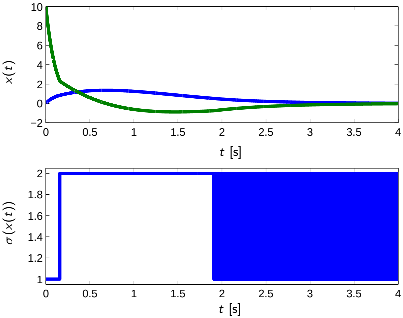
- By numerical simulation we have determined the actual cost given by

$$J_2(\sigma_{so}) = 1.6357$$

We can conclude that the min-type switching rule $\sigma(\cdot)$ is strictly consistent with a cost reduction of 40% !

Example 3 - $\mathcal{H}_2$ performance

- The state trajectories and the switching rule.



Notice the existence of stable sliding modes !

Example 4 - $\mathcal{H}_\infty$ performance

- Consider a switched linear system composed of two stable subsystems

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -9 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \; H = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; G_1 = G_2 = 1$$

we can calculate

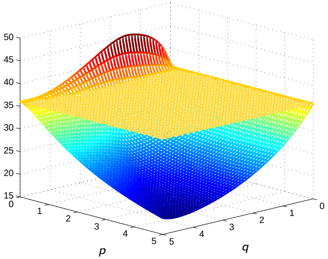
$$\begin{aligned} J_\infty(\sigma) &= \min_{\ell \in \{1,2\}} \| E_\ell (sI - A_\ell)^{-1} H_\ell + G \|_\infty^2 \\ &= \min\{36.0463 \, , \; \underbrace{35.9356}_{\sigma(t)=2, \; \forall t \geq 0} \} \end{aligned}$$

Example 4 - $\mathcal{H}_\infty$ performance

- Adopting a Metzler matrix of the form

$$\Pi = \begin{bmatrix} -p & q \\ p & -q \end{bmatrix}$$

we have determined the minimum guaranteed cost for all (p, q) inside the box $[0, 2] \times [0, 2]$ as shown in the next figure where the plane surface concerns $\min_{\sigma \in \mathcal{C}} J_\infty(\sigma)$.



Example 4 - $\mathcal{H}_\infty$ performance

- The best guaranteed cost was obtained for

$$\Pi^* \approx \begin{bmatrix} -5.0 & 4.5 \\ 5.0 & -4.5 \end{bmatrix} \Rightarrow J_\infty^{so} = 18.0677$$

- The obtained cost was

$$J_\infty(\sigma_{so}) < \underbrace{18.0677}_{J_\infty^{so}} < \min_{\sigma \in \mathcal{C}} J_\infty(\sigma) = 35.9356$$

We can conclude that the min-type switching rule $\sigma(\cdot)$ is strictly consistent with a cost reduction of at least 50% !

State feedback control design

- The idea now is to generalize the previous $\mathcal{H}_2$ and $\mathcal{H}_\infty$ conditions to deal with the continuous-time system

$$
\begin{aligned}
\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + H_{\sigma(t)}w(t), \; x(0) = 0 \\
z(t) &= E_{\sigma(t)}x(t) + F_{\sigma(t)}u + G_{\sigma(t)}w(t)
\end{aligned}
$$

where the control law

$$u(t) = K_{\sigma(x(t))}x(t)$$

must be designed together with the switching rule $\sigma(x)$ in order to preserve stability and $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance.

- Connecting u to the system, we obtain the closed loop system

$$
\begin{aligned}
\dot{x}(t) &= (A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)})x + H_{\sigma(t)}w(t), \; x(0) = 0 \\
z(t) &= (E_{\sigma(t)} + F_{\sigma(t)}K_{\sigma(t)})x(t) + G_{\sigma(t)}w(t)
\end{aligned}
$$

State feedback control

Defining $\mathrm{He}\{X\} = X + X'$ we have :

Theorem : $\mathcal{H}_2$ control

If there exist symmetric matrices S_i, T_{ij}, matrices Y_i and a Metzler matrix $\Pi \in \mathcal{M}_c$ satisfying the Lyapunov-Meztler inequalities

$$\begin{bmatrix} \mathrm{He}\{A_i S_i + B_i Y_i\} + \sum_{j \neq i=1}^{N} \pi_{ji} T_{ij} & \bullet \\ E_i S_i + F_i Y_i & -I \end{bmatrix} < 0, \ \ i \in \mathbb{K}$$

$$\begin{bmatrix} T_{ij} + S_i & \bullet \\ S_i & S_j \end{bmatrix} > 0, \ \ i \neq j \in \mathbb{K} \times \mathbb{K}$$

then the switching rule $\sigma(x) = \arg\min_{i \in \mathbb{K}} x' S_i^{-1} x$ and the state feedback gains $K_i = Y_i S_i^{-1}$ assure the global asymptotic stability of the origin and satisfies

$$J_2(\sigma) < \min_{i \in \mathbb{K}} \mathrm{Tr}(H_i' S_i^{-1} H_i)$$

State feedback control

Theorem : $\mathcal{H}_\infty$ control

If there exist symmetric matrices S_i, T_{ij}, matrices Y_i and a scalar $\rho > 0$ and a Metzler matrix $\Pi \in \mathcal{M}_c$ satisfying the Riccati-Meztler inequalities

$$\begin{bmatrix} \mathrm{H_e}\{A_i S_i + B_i Y_i\} + \sum_{j \neq i=1}^{N} \pi_{ji} T_{ij} & \bullet & \bullet \\ H_i' & -\rho I & \bullet \\ E_i S_i + F_i Y_i & G_i & -I \end{bmatrix} < 0(*), \ i \in \mathbb{K}$$

$$\begin{bmatrix} T_{ij} + S_i & \bullet \\ S_i & S_j \end{bmatrix} > 0, \ i \neq j \in \mathbb{K} \times \mathbb{K}$$

then the switching rule $\sigma(x) = \arg\min_{i \in \mathbb{K}} x' S_i^{-1} x$ and the state feedback gains $K_i = Y_i S_i^{-1}$ assure the global asymptotic stability of the origin and satisfies $J_\infty(\sigma) < \rho$.

State feedback control

- Both conditions were obtained from the from the fact that $T_{ij} > S_i S_j^{-1} S_i - S_i$ for all $i \neq j$ which provides

$$\sum_{j \neq i=1}^{N} \pi_{ji} T_{ij} > \sum_{j \neq i=1}^{N} \pi_{ji} \left(S_i S_j^{-1} S_i - S_i S_i^{-1} S_i \right)$$
$$> \sum_{j=1}^{N} \pi_{ji} S_i S_j^{-1} S_i$$

- For the $\mathcal{H}_\infty$ case, considering this inequality and multiplying both sides of (*) by $\mathrm{diag}\{S_i^{-1}, I, I\}$, we obtain the original Riccati-Metzler inequalities after performing the Schur Complement with respect to the last row and column and making the replacements $A_i \to A_i + B_i K_i$ and $E_i \to E_i + F_i K_i$.
- Similar procedure can be made in the $\mathcal{H}_2$ case.

Problems

Consider the switched linear system

$$\dot{x} = A_\sigma x, \;\; x(0) = x_0$$
$$z = E_\sigma x$$

1) Adopting the min-type Lyapunov-function

$$v(x) = \min_{i \in \mathbb{K}} x' P_i x$$

- Find the conditions that assure stability for an arbitrary switching rule $\sigma(t)$.
- Do the obtained conditions require some stability property of each isolated subsystem ?
- Show that the obtained conditions contain the quadratic ones $A_i'P + PA_i + E_i'E_i < 0, \;\; \forall i \in \mathbb{K}$, as particular case.

Problems

2) Is it possible to assure global asymptotic stability by adopting the max-type Lyapunov function

$$V(x) = \max_{i \in \mathbb{K}} x' P_i x$$

associated with the switching function

$$\sigma(x) = \arg \max_{i \in \mathbb{K}} x' P_i x$$

- If the answer is positive, present the stability conditions.
- If negative, justify mathematically.

3) Show that the modified Lyapunov-Metzler inequalities are indeed a particular case of the original ones.

Problems

4) For the switched linear system defined by matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}, \; x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$E_1 = E_2 = I$. Elaborate a Matlab program to solve

$$\min_{i \in \mathbb{K}} \inf_{P_i > 0} x_0' P_i x_0$$

subject to the Lyapunov-Metzler inequalities Theorem : Stability (pag 16).

- Provide P_1, P_2, Π and the guaranteed cost for a generic $\Pi \in \mathcal{M}_c$.
- Provide P_1, P_2, Π and the guaranteed cost for $\Pi \in \mathcal{M}_c$ with the same main diagonals.
- Provide P_1, P_2, Π and the guaranteed cost for $\Pi = -I + \nu[1 \;\; 1] \in \mathcal{M}_c, \; \nu \in \Lambda$.
- Compare the results.

Problems

5) For the same switched linear system, solve the problem

$$\inf_{P>0} x_0' P x_0$$

for the conditions of Lemma : Quadratic stability (pag 11) by searching inside the simplex $\lambda \in \Lambda$. Provide the solution P, $\lambda \in \Lambda$ and compare the result with Problem 3).

6) Implement the switching rule of Problem 4) for the generic $\Pi \in \mathcal{M}_c$ and the switching rule of Problem 5) and show that in both cases the state trajectories converge indeed to the origin.

Problems

7) For the more general switched linear system

$$\dot{x} = A_\sigma x + Hw, \;\; x(0) = 0$$
$$z = Ex + Gw$$

Define $H_{wz}(s) = E(sI - A_\lambda)^{-1}H + G$, $\lambda \in \Lambda$, show that :

a) Considering $G = 0$, the norm $\|H_{wz}(s)\|_2^2$ is an upper bound for the $\mathcal{H}_2$ performance index $J_2(\sigma)$. Obtain the conditions as a particular case of the Lyapunov-Metzler inequality.

b) For the previous item, find the corresponding stabilizing state dependent switching function $\sigma(x)$.

c) The norm $\|H_{wz}(s)\|_\infty^2$ is an upper bound for the $\mathcal{H}_\infty$ performance index $J_\infty(\sigma)$. Obtain the associated conditions based on an unique matrix $P > 0$.

d) For the previous item, find the corresponding state-input dependent stabilizing switching function $\sigma(x, w)$.

e) For item c), find the corresponding state dependent stabilizing switching function $\sigma(x)$.

Problems

8) Concerning the previous problem, is it possible to associate a stabilizing switching function $\sigma(x)$ with the norm of $\|E(sI - A_\lambda)^{-1}H_\lambda + G\|_\infty^2$ and what about $\sigma(x, w)$?
9) Consider the system of Problem #7 with

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix}, \; H = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \; E' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, G = 0$$

Elaborate a Matlab program to find $\lambda \in \Lambda$ is order to obtain :

a) The smallest $\|H_{wz}(s)\|_2^2$.
b) Implement the correspondent switching function $\sigma(x)$, show that the state trajectories converge indeed to the origin and, by numerical simulation, determine $\|z\|_2^2$.
c) Solve the Lyapunov-Metzler inequalities with $\Pi \in \mathcal{M}_c$ and provide Π, P_1, P_2 important to implement the switching function $\sigma(x) = \arg\min_{i \in \mathbb{K}} x' P_i x$.

Problems

d) Show that the state trajectories converge indeed to the origin and, by numerical simulation, provide $\|z\|_2^2$.

e) Compare the costs obtained in the itens a), b), c), and d).

10) Consider the system of pag 59 with matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -9 & 5 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & -7 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; H = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \; E' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; F = 1, \; G = 0$$

with $H_i = H$, $E_i = E$, $B_i = B$, $F_i = F$, $G_i = G$ for all $i = 1, 2$.

a) For each isolated subsystem, find the gains $K_i, \; i = \{1, 2\}$ that minimizes the $\mathcal{H}_2$ norm and present the correspondent norm.

Problems

b) Using the gains determined in the previous item, solve the conditions of Theorem : $\mathcal{H}_2$ performance (pag 41) for the closed-loop system. Provide the state trajectories, the cost J_2^{so} and the solution P_1, P_2, Π. Compare J_2^{so} with the norms of each subsystem based on the concept of consistency.

c) Solve the conditions of Theorem : $\mathcal{H}_2$ control (pag 60). Provide the state trajectories, the cost J_2^{so} and the solution P_1, P_2, K_1, K_2 and Π. Compare the cost obtained with item b).

Problems

11) Consider the LPV system

$$\Sigma(\lambda) := \begin{cases} \dot{x}(t) = A_{\lambda(t)}x(t) + B_{\lambda(t)}u(t) + Hw(t) \\ z(t) = C_{\sigma(x(t))}x + D_{\sigma(x(t))}u(t) \end{cases}$$

where $\lambda(t) \in \Lambda$ is a time-varying uncertain parameter. Based on the Lyapunov-Metzler inequalities with a parameter-dependent Metzler matrix $\Pi(\lambda) \in \mathcal{M}_c$ defined as

$$\pi_{ji}(\lambda) := \begin{cases} \gamma_i \lambda_j, & j \neq i \\ \gamma_i(\lambda_i - 1), & j = i \end{cases}$$

find the conditions for which the control law

$$u(t) = K_{\sigma(x(t))}x(t)$$

assures global asymptotic stability of the equilibrium point and an $\mathcal{H}_2$ guaranteed cost.