Gain-scheduled $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control of discrete-time polytopic time-varying systems

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Abstract: This study presents $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance analysis and synthesis procedures for the design of both gain-scheduled and robust static output feedback controllers for discrete-time linear systems with time-varying parameters. The obtained controllers guarantee an upper bound on the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance of the closed-loop system. As an immediate extension, the mixed $\mathcal{H}_2$/ $\mathcal{H}_\infty$ guaranteed cost control problem is also addressed. The scheduling parameters vary inside a polytope and are assumed to be a priori unknown, but measured in real-time. If bounds on the rate of parameter variation are known, they can be taken into account, providing less conservative results. The geometric properties of the polytopic domain are exploited to derive finite sets of linear matrix inequalities (LMIs) based on the existence of a parameter-dependent Lyapunov function. An application of the methodology to a realistic vibroacoustic problem, with experimentally obtained data, illustrates the benefits of the proposed approach and shows that the techniques can be used for real engineering problems.

1 Introduction

For more than a decade, both analysis and synthesis techniques for linear parameter-varying (LPV) systems have received a lot of attention from the control community. This stems from the fact that LPV models are useful to describe the dynamics of linear systems affected by time-varying parameters as well as to represent non-linear systems in terms of a family of linear models [1]. In the LPV control framework, the scheduling parameters that govern the variation of the dynamics of the system are usually a priori unknown, but measured or estimated in real-time [2]. There is a continuing effort towards the design of LPV controllers, scheduled as a function of the varying parameters, to achieve higher performance while still guaranteeing stability for all possible parameter variations (see, for instance, [1, 3–8]).

One practical way (e.g. [9–11]) to compute a gain-scheduled controller for a given LPV system consists of the following steps. First, determine a family of linear time-invariant (LTI) models by selecting different operating conditions of the system and then design a local controller for each one of the LTI models. Next, based on the values of the parameters (measured or estimated on-line), schedule the local controllers using some interpolation method. The final step consists of checking the closed-loop stability and performance using extensive simulation. Although the system performance can be improved by means of increasing the number of local models (at the expense of increasing the computational burden) this approach may be unreliable, since the closed-loop stability and performance are only verified through simulations.

To overcome these difficulties, several results for analysis and synthesis techniques for LPV systems have been proposed based on different types of Lyapunov functions that are able to guarantee closed-loop stability and performance. However, many of these approaches (see, for example, [12–14]) use the notion of quadratic stability
where the Lyapunov matrix is assumed constant. This generally leads to conservative results for practical applications since it allows arbitrarily fast variation of the scheduling parameters.

To mitigate some of the conservatism associated with the quadratic stability-based approaches, many works using parameter-dependent Lyapunov functions have been published. For discrete-time linear time-varying (LTV) systems with arbitrarily fast varying parameters, for instance, the design of stabilising controllers has been presented in [15] using a polytopic structure for the Lyapunov matrix and in [16] using a piecewise Lyapunov function. Oliveira and Peres [17] provide a synthesis procedure for stabilising controllers for discrete-time LTV systems, when an a priori bound on the rate of parameter variation is known, using a Lyapunov matrix with a polytopic structure. Regarding the design of controllers that ensure $H_\infty$ performance, results for continuous-time LTV systems have been presented in [18, 19]. In [18] a parameter-dependent weighted sum of Lyapunov matrices is used. On the other hand, homogeneous polynomially parameter-dependent Lyapunov functions are used in [19]. For the discrete-time case, Amato et al. [20] use a piecewise constant Lyapunov matrix and Montagner et al. [21] use a polytopic structure for the Lyapunov matrix. A bound on the rate of variation of the parameters is considered in [19] for the continuous-time case and in [20] for the discrete-time case. For the $H_2$ performance case, control design methods for continuous-time systems have been developed in [22] using an affine Lyapunov matrix and in [23] using a quadratic Lyapunov matrix. In [24], synthesis conditions are provided for homogeneous polynomially parameter-dependent controllers under quadratic stability. In particular, the works [22, 23] consider bounds on the rate of variation. To the best of the authors’ knowledge, no methods have been published that use parameter-dependent Lyapunov functions for the design of $H_2$ controllers for discrete-time LTV systems.

The aim of this paper is to provide linear matrix inequality (LMI) conditions for the synthesis of gain-scheduled static output feedback controllers for discrete-time linear systems with time-varying parameters belonging to a polytope with a prescribed bound on the rate of parameter variation. Using a time-varying analogy of the result presented in [25], a suboptimal multiobjective $H_2/H_\infty$ control design problem can be conveniently solved. It is worth to emphasise that this paper extends preliminary results published in [26, 27] by mainly providing a new and more realistic model for the uncertainty domain, a throughout theoretical background of the $H_2$ performance as described in Section 3 and proved in the Appendix, multiobjective gain-scheduled $H_2$ and $H_\infty$ control synthesis conditions and a realistic engineering application with an LPV model obtained from experimental data.

This paper is organised as follows. Section 3 presents general theoretical background regarding $H_2$ and $H_\infty$ performance of discrete-time LTV systems. Section 4 introduces some preliminaries with respect to the time variation of the scheduling parameters and then applies the machinery of Section 3 to the specific case of polytopic discrete-time LTV systems with known bounds on the rate of the parameter variation. Section 5 extends the analysis results and presents synthesis procedures for both gain-scheduled and robust static output feedback controllers. These synthesis procedures are applied to a numerical example in Section 6 and to a vibroacoustic problem with realistic numerical data in Section 7. Conclusions and final remarks follow in Section 8.

## 2 Notation

The set of real numbers is denoted as $\mathbb{R}$, the set of integer numbers as $\mathbb{Z}$ and the set of non-negative integers as $\mathbb{Z}_+$. The symbol $(\cdot)^T$ indicates transpose. The $\ell_2^2$ space of square-summable sequences on $\mathbb{Z}_+$ is given by

$$
\ell_2^2 \triangleq \left\{ f: \mathbb{Z}_+ \to \mathbb{R}^n \mid \sum_{k=0}^{\infty} f(k)^T f(k) < \infty \right\}
$$

The corresponding 2-norm is defined as $\|x(k)\|_2^2 = \sum_{k=0}^{\infty} x(k)^T x(k)$. The expectation operator is denoted by $\mathbb{E}$. The identity matrix of size $r \times r$ is denoted as $I_r$. The notation $\Theta_{n,m}$ indicates an $n \times m$ matrix of zeros. The Kronecker delta $\delta_{kl}$ is 1 for $k = l$ and 0 otherwise. The convex hull of a set $X$ is denoted by $\mathbb{C}X$.

## 3 Performance of discrete-time LTV systems

This section introduces preliminary results concerning the characterisation of $H_2$ and $H_\infty$ guaranteed performance of discrete-time LTV systems. Although some material can be found in the literature, it is clarifying to restate some main results covering the characterisation of the $H_2$ performance for LTV systems, since only an outline is found in [28] for the continuous-time case and in [29] for the discrete-time case.

The results presented in this section are valid for general discrete-time LTV systems. Later, in Section 4, the results are particularised for the class of discrete-time polytopic LTV systems with bounds on the rate of parameter variation, providing finite sets of convex conditions that guarantee an upper bound on the $H_2$ and $H_\infty$ performance.

The discrete-time LTV system $H$ considered in this section is assumed to have a finite-dimensional state-space realisation

$$
H := \begin{cases} 
\begin{align*}
x(k+1) &= A(k)x(k) + B_{u0}(k)\omega(k), \\
z(k) &= C_{x}(k)x(k) + D_{u}(k)\omega(k)
\end{align*}
\end{cases}
$$

where $x(k) \in \mathbb{R}^n$ is the state, $\omega(k) \in \mathbb{R}^r$ the exogenous input,
\(z(k) \in \mathbb{R}^p\) the system output, and \(A(k) \in \mathbb{R}^{n \times n}\), \(B_w(k) \in \mathbb{R}^{n \times r}\), \(C_e(k) \in \mathbb{R}^{r \times n}\) and \(D_w(k) \in \mathbb{R}^{r \times r}\) the time-varying system matrices. It is assumed that all matrices are real, bounded, and defined for \(k \geq 0\).

### 3.1 \(H_\infty\) Performance

The \(H_\infty\) performance of system \(H\), given by (1), is defined by the quantity

\[
\|H\|_\infty = \sup_{\|u(k)\|_2 \neq 0} \frac{\|z(k)\|_2}{\|u(k)\|_2}
\]

with \(w(k) \in \ell_2^2\) and \(z(k) \in \ell_2^2\). Based on the bounded real lemma, an upper bound for the \(H_\infty\) performance can be characterised by the following theorem, as shown in \([30]\).

**Theorem 1:** Consider system \(H\) given by (1). If there exist bounded matrices \(G(k)\) and \(P(k) = P(k)^T > 0\) for all \(k \geq 0\) such that

\[
\begin{bmatrix}
P(k+1) & A(k)G(k) & B_w(k) & 0 \\
G(k)^TA(k)^T & G(k) + G(k)^T - P(k) & 0 & G(k)^TC_e(k)^T \\
B_w(k)^T & 0 & \eta I & D_w(k)^T \\
0 & C_e(k)G(k) & D_w(k) & \eta I
\end{bmatrix} > 0
\]

then the system \(H\) is exponentially stable and

\[
\|H\|_\infty \leq \inf_{P(k), G(k), \eta} \eta
\]

### 3.2 \(H_2\) Performance

This section introduces the \(H_2\) performance for LTV systems. First, recall that for discrete-time LTI systems the following three main definitions (e.g. \([31, 32]\)) are usually employed to characterise the \(H_2\)-norm:

1. If \(H(z)\) represents the transfer function of an LTI system \(H\), then its \(H_2\)-norm can be defined as

\[
\|H(z)\|_2 := \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}(H(e^{j\omega})^T H(e^{j\omega})) d\omega
\]

2. If \(\{e_1, \ldots, e_r\}\) is a basis for the input space and \(z_k(\cdot)\) is the output of an LTI system \(H\) when an impulse \(\delta(k)e_i\) is applied, then its \(H_2\)-norm can be defined as

\[
\|H\|_2 := \sum_{i=1}^r \|z_i(\cdot)\|_2^2
\]

3. If \(z(\cdot)\) is the output of an LTI system when a zero-mean white noise Gaussian process \(w(\cdot)\) with identity covariance matrix is applied, then its \(H_2\)-norm can be defined as

\[
\|H\|_2 := \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} z(k)^T z(k)
\]

where the positive integer \(T\) denotes the time horizon.

The first definition cannot be extended to time-varying systems, since, in this case, the notion of transfer function is not well defined. On the other hand, definitions 2 and 3 can be extended to time-varying systems. In general, these two definitions will not provide the same values. Moreover, in definition 2, the computation of the norm can depend on the basis chosen for the input space. Therefore, in this paper, the \(H_2\) performance of a discrete-time LTV system is defined using definition 3 as done in \([28, 29, 31]\).

**Definition 1:** Infinite horizon \(H_2\) performance of a discrete-time LTV system.

Suppose that the system \(H\), given by (1), is exponentially stable, then its infinite horizon \(H_2\) performance is defined by

\[
\|H\|_2 := \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} z(k)^T z(k)
\]

when the system input \(w(k)\) is a zero-mean white-noise Gaussian process with identity covariance matrix.

The following lemma provides a characterisation for the infinite horizon \(H_2\) performance.

**Lemma 1:** Suppose that the system \(H\), given by (1), is exponentially stable, then its infinite horizon \(H_2\) performance can be characterised as

\[
\|H\|_2^2 = \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \text{Tr}(C_e(k)\bar{P}(k)C_e(k)^T + D_w(k)D_w(k)^T) + \text{Tr}(\bar{Q}(k+1)B_w(k))
\]

in which \(\bar{P}(k)\) is the controllability Gramian satisfying

\[
\bar{P}(k+1) = A(k)\bar{P}(k)A(k)^T + B_w(k)B_w(k)^T, \quad \bar{P}(0) = 0
\]

and \(\bar{Q}(k)\) is the observability Gramian satisfying

\[
\bar{Q}(k) = A(k)^T\bar{Q}(k+1)A(k) + C_e(k)^TC_e(k), \quad \lim_{k \to \infty} \bar{Q}(k) = 0
\]

The proof of Lemma 1 is presented in Appendix 11.1. The
Theorem 2: Consider system $H$ given by (1). The following statements are equivalent.

(a) If there exist bounded matrices $P(k) = P(k)^T > 0$ and $W(k) = W(k)^T > 0$ for all $k \geq 0$ such that
\[
\begin{bmatrix}
P(k + 1) - A(k)P(k)A(k)^T & B_w(k)^T \\
B_w(k) & I
\end{bmatrix} > 0
\]  
and
\[
\begin{bmatrix}
W(k) - D_w(k)D_w(k)^T & C_z(k)P(k) \\
P(k)C_z(k)^T & P(k)
\end{bmatrix} > 0
\]
then the system $H$ is exponentially stable and
\[
\|H\|_2^2 \leq \inf_{P(k),W(k)} \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \text{Tr}[W(k)]
\]

(b) If there exist bounded matrices $Q(k) = Q(k)^T > 0$ and $S(k) = S(k)^T > 0$ for all $k \geq 0$ such that
\[
\begin{bmatrix}
Q(k) - A(k)^T Q(k + 1)A(k) & C_z(k)^T \\
C_z(k) & I
\end{bmatrix} > 0
\]
and
\[
\begin{bmatrix}
S(k) - D_w(k)^T D_w(k) & B_w(k)^T Q(k + 1) \\
Q(k + 1)B_w(k) & Q(k + 1)
\end{bmatrix} > 0
\]
then the system $H$ is exponentially stable and
\[
\|H\|_2^2 \leq \inf_{Q(k),S(k)} \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \text{Tr}[S(k)]
\]

The proof is presented in Appendix 11.2. It is worth to emphasise that $V_P(x(k), k) = x(k)^T P(k)^{-1} x(k)$ and $V_Q(x(k), k) = x(k)^T Q(k)x(k)$, with $P()$ and $Q()$ given above, are Lyapunov functions for the system $H$.

The characterisation of the $\mathcal{H}_2$ performance, presented in Theorem 2(a), can be extended by introducing additional instrumental variables as follows.

Theorem 3: Consider system $H$ given by (1). If there exist bounded matrices $G(k), P(k) = P(k)^T > 0$ and $W(k) = W(k)^T > 0$ for all $k \geq 0$ such that
\[
\begin{bmatrix}
P(k + 1) - A(k)G(k)A(k)^T & B_w(k)^T \\
G(k)^T A(k)^T & G(k)^T + G(k)^T - P(k) \\
B_w(k) & 0 \\
0 & I
\end{bmatrix} > 0
\]
and
\[
\begin{bmatrix}
W(k) - D_w(k)D_w(k)^T & C_z(k)G(k) \\
G(k)^T C_z(k)^T & G(k) + G(k)^T - P(k)
\end{bmatrix} > 0
\]
then the system $H$ is exponentially stable and
\[
\|H\|_2^2 \leq \inf_{P(k),G(k),W(k)} \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \text{Tr}[W(k)]
\]

Similar results can be established with respect to the conditions in Theorem 2(b). The equivalence between the LMIs of Theorems 2(a) and 3 can be easily shown following the steps of the proof for Theorem 1 in [25] for the time-invariant case.

4 Performance of discrete-time polytopic LTV systems

In this section, Theorems 1 and 3 that introduced the extended characterisation of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of general discrete-time LTV systems are particularised for the specific case of polytopic LTV systems. For this class of systems, a finite set of LMIs is provided, defined in the vertices of the polytope, that guarantees an upper bound on the performance of the polytopic LTV system. Bounds on the rate of variation of the scheduling parameter are also considered. The modelling of the uncertainty domain is first presented; afterwards, the finite sets of LMIs that guarantee an $\mathcal{H}_\infty$ and $\mathcal{H}_2$ upper bound on the system performance are introduced.

4.1 Modelling of the uncertainty domain

Consider the polytopic time-varying discrete-time linear system
\[
H := \left\{ x(k+1) = A(x(k))x(k) + B_u(x(k))w(k) + B_w(x(k))u(k) \\
z(k) = C_z(x(k))x(k) + D_w(x(k))w(k) + D_u(x(k))u(k) \right\}
\]
where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^r$ the exogenous input, $u(k) \in \mathbb{R}^m$ the control input and $z(k) \in \mathbb{R}^p$ the system output. The system matrices $A(x(k)) \in \mathbb{R}^{n \times n}, B_u(x(k)) \in \mathbb{R}^{n \times r}, B_w(x(k)) \in \mathbb{R}^{n \times m}, C_z(x(k)) \in \mathbb{R}^{p \times n}, D_u(x(k)) \in \mathbb{R}^{p \times r}$ and $D_w(x(k)) \in \mathbb{R}^{p \times r}$ belong to the polytope
\[
\mathcal{S} = \left\{ (A, B_w, B_u, C_z, D_w, D_u)(\alpha(k)) : \\
(A, B_w, B_u, C_z, D_w, D_u)(\alpha(k)) \right\}
\]
and
\[
\mathcal{D} = \left\{ \alpha(k)(A_1, B_{w1}, B_{u1}, C_{z1}, D_{w1}, D_{u1}), \alpha(k) \in \Lambda_N \right\}
\]
with \( A_i, B_{u,i}, C_{x,i}, D_{y,i} \) the vertices of the polytope and \( \alpha(\dot{k}) \in \mathbb{R}^n \) the vector of time-varying parameters lying in the unit simplex

\[
\Lambda_N = \left\{ \xi \in \mathbb{R}^N : \sum_{i=1}^N \xi_i = 1, \xi_i \geq 0, \quad i = 1, \ldots, N \right\}
\]  

(18)

The rate of variation of the parameters

\[
\Delta \alpha_i(\dot{k}) = \alpha_i(k+1) - \alpha_i(k), \quad i = 1, \ldots, N
\]  

(19)

is assumed to be limited by an a priori known bound \( b \in \mathbb{R} \) such that

\[
-b \leq \Delta \alpha_i(\dot{k}) \leq b, \quad i = 1, \ldots, N
\]  

(20)

with \( b \in [0,1] \). Since \( \alpha(\dot{k}) \in \Lambda_N \), it is clear from (19) that

\[
\sum_{i=1}^N \Delta \alpha_i(\dot{k}) = 0
\]  

(21)

The uncertainty domain where the vector \( (\alpha(\dot{k}), \Delta \alpha(\dot{k}))^T \in \mathbb{R}^{2N} \) assumes values can be modelled by the compact set

\[
\Gamma_b = \left\{ \delta \in \mathbb{R}^{2N}, \delta \in \alpha g^1, \ldots, g^M \right\},
\]

\[
g^j = \left( \begin{array}{c} f^j \\ b^j \end{array} \right), \quad f^j \in \mathbb{R}^N, \quad b^j \in \mathbb{R}^N,
\]

\[
\sum_{i=1}^N f^j_i = 1 \text{ with } f^j_i \geq 0, \quad i = 1, \ldots, N,
\]

\[
\sum_{i=1}^N b^j_i = 0, \quad j = 1, \ldots, M
\]  

(22)

defined as the convex combination of the vectors \( g^j \), for \( j = 1, \ldots, M \), given a priori. Notice that this definition of \( \Gamma_b \) ensures that \( \alpha(\dot{k}) \in \Lambda_N \) and that (21) holds for all \( k \geq 0 \). The vectors \( g^j \in \Gamma_b \), for \( j = 1, \ldots, M \), can be constructed in a systematic way for a given \( b \) by searching for all possible solutions of the equalities \( \sum_{i=1}^N \alpha_i = 1 \) and \( \sum_{i=1}^N \Delta \alpha_i = 0 \) using the extreme points of the constraints given in (20); for \( i = 1, \ldots, N \). However, taking into account just (20) in the modelling of the uncertainty domain introduces conservatism since the bounds on \( \Delta \alpha_i(\dot{k}) \) are considered independent of \( \alpha_i(\dot{k}) \) while in fact, they are highly dependent of the value of \( \alpha_i(\dot{k}) \), as illustrated in Fig. 1. The light grey area indicates the region in the \((\alpha_i, \Delta \alpha_i)-\)space where \( \Delta \alpha_i \) can assume values as a function of \( \alpha_i \) (indicated in light grey).

**Figure 1** Area in the \((\alpha_i, \Delta \alpha_i)-\)space where \( \Delta \alpha_i \) can assume values as a function of \( \alpha_i \) (indicated in light grey)

The dark grey areas are unreachable since \( \alpha \in \Lambda_N \).

In the modelling of the uncertainty domain, the dark grey areas are also taken into account, thus producing conservative results. Therefore to find the vectors \( g^j \), the solutions of \( \sum_{i=1}^N \alpha_i = 1 \) and \( \sum_{i=1}^N \Delta \alpha_i = 0 \) need to be sought using the vertices (23) of the feasible region in the \((\alpha_i, \Delta \alpha_i)-\)space, for \( i = 1, \ldots, N \).

As an illustration, consider \( \alpha \in \Lambda_2 \). In this case, the vectors \( g^j \), for \( j = 1, \ldots, M \), can be found to be

\[
\begin{bmatrix} g^1 & g^2 & \cdots & g^M \end{bmatrix} = \begin{bmatrix} f^1 & f^2 & \cdots & f^M \\
1 & 1 & 0 & \cdots & 0 & b & 1 - b \\
1 & 0 & b & \cdots & -b & 1 - b \\
0 & 0 & b & \cdots & -b & -b \\
0 & -b & 0 & \cdots & b & b \\
0 & b & 0 & \cdots & b & b \\
\end{bmatrix}
\]

(23)

and \( M = 6 \). For a polytopic system (16) with system matrices varying in the polytope (17) with \( N \) vertices and a given value of the bound \( b \), Appendix 11.3 shows how to generate the vertices (23) of the uncertainty set \( \Gamma_b \). Once the columns of the set \( \Gamma_b \) are defined, the convex characterisation

\[
(\alpha(\dot{k}), \Delta \alpha(\dot{k}))^T = \sum_{j=1}^M \begin{bmatrix} f^j \\ b^j \end{bmatrix} \gamma_j(\dot{k})
\]  

(24)

using \( \gamma(\dot{k}) \in \Lambda_M \) can be exploited in the derivations of the LMI conditions.

### 4.2 \( H_\infty \) Performance

The LMI condition in Theorem 1 is now particularised for the polytopic LTV systems, described by (16).

**Theorem 4:** Consider system \( H \) given by (16). If there exist parameter-dependent matrices \( G(\alpha(\dot{k})) \) and
Then the system $H$ is exponentially stable and

\[ \| H \|_\infty \leq \inf_{P(\alpha(\cdot)), G(\alpha(\cdot), \eta)} \eta \]

This LMI condition follows directly from (2) in Theorem 1 by considering the specific time dependency of system (16) on the time-varying parameter $\alpha(\cdot)$.

It is worth to emphasize that the conditions of Theorem 4, which consist of evaluating the parameter-dependent LMI for all $\alpha(\cdot)$ in the unit simplex $\Lambda_N$, lead to an infinite dimensional problem. However, a finite-dimensional set of LMI conditions can be obtained by imposing some particular structure on the Lyapunov matrix $P(\alpha(\cdot))$ and the slack variable $G(\alpha(\cdot))$. Choosing $P(\alpha(\cdot))$ to have the following affine parameter-dependent structure

\[ P(\alpha(\cdot)) = \sum_{i=1}^{N} \alpha_i(\cdot) P_i, \quad \alpha_i(\cdot) \in \Lambda_N \quad (26) \]

it can be shown, using (24), that

\[ P(\alpha(\cdot)) = \sum_{i=1}^{N} \alpha_i(\cdot) P_i = \sum_{i=1}^{N} \left( \sum_{j=1}^{M} f^j_i(\cdot) \gamma_j(\cdot) \right) P_i \]

\[ = \sum_{j=1}^{M} \gamma_j(\cdot) \left( \sum_{i=1}^{N} f^j_i(\cdot) P_i \right) = \sum_{j=1}^{M} \gamma_j(\cdot) \hat{P}_j = \hat{P}(\gamma(\cdot)) \quad (27) \]

with $\hat{P}_j = \sum_{i=1}^{N} f^j_i(\cdot) P_i$. Using the same structure for $\alpha(\cdot)$, all system matrices in (16) can also be converted to a new representation in terms of $\gamma(\cdot) \in \Lambda_M$. For instance, $A(\alpha(\cdot))$ becomes

\[ A(\alpha(\cdot)) = \hat{A}(\gamma(\cdot)) = \sum_{j=1}^{M} \gamma_j(\cdot) \hat{A}_j \quad (28) \]

with $\hat{A}_j = \sum_{i=1}^{N} f^j_i(\cdot) \hat{A}_i$. The other system matrices can be converted similarly. Moreover, combining (24) and the fact that $\alpha(k+1) = \alpha(k) + \Delta \alpha(k)$, it follows that

\[ P(\alpha(k+1)) = \sum_{i=1}^{N} (\alpha_i(k) + \Delta \alpha_i(k)) P_i \]

\[ = \sum_{i=1}^{N} \left( \sum_{j=1}^{M} (f^j_i(\cdot) + \delta_j(\cdot)) \gamma_j(\cdot) \right) P_i \]

\[ = \sum_{j=1}^{M} \gamma_j(\cdot) \left( \sum_{i=1}^{N} (f^j_i(\cdot) + \delta_j(\cdot)) P_i \right) \]

\[ = \sum_{j=1}^{M} \gamma_j(\cdot) \hat{P}_j = \hat{P}(\gamma(\cdot)) \quad (29) \]

with $\hat{P}_j = \sum_{i=1}^{N} (f^j_i(\cdot) + \delta_j(\cdot)) P_i$. As a result of this new representation, the LMI (25) from Theorem 4 can be rewritten with a dependency on $\gamma(\cdot)$. Consequently, a convenient choice for the parameterisation of the slack variable is given by

\[ G(\gamma(\cdot)) = \sum_{j=1}^{M} \gamma_j(\cdot) G_j, \quad \gamma(\cdot) \in \Lambda_M \quad (30) \]

Using the parameterisations (26)–(30), the next theorem presents a finite-dimensional set of LMIs that guarantees the LMI condition of Theorem 4.

**Theorem 5:** Consider system $H$, given by (16). Assume that the vectors $f^j$ and $b^j$ of $\Gamma_a$ are given. If there exist, for $j = 1, \ldots, M$, matrices $G_j \in \mathbb{R}^{n \times n}$ and, for $i = 1, \ldots, N$, symmetric positive-definite matrices $P_i \in \mathbb{R}^{n \times n}$ such that

\[
\begin{bmatrix}
    P(\alpha(k+1)) & \star & \star \\
    G(\alpha(\cdot))^T A(\alpha(\cdot))^T & G(\alpha(\cdot)) + G(\alpha(\cdot))^T - P(\alpha(\cdot)) & \eta I \\
    0 & C(\alpha(\cdot))^T G(\alpha(\cdot)) & D(\alpha(\cdot)) - \eta I \\
\end{bmatrix} = \Theta > 0 \quad (25)
\]

and

\[
\begin{bmatrix}
    \sum_{i=1}^{N} (f^j_i + b_i^j) P_i & \star & \star & \star \\
    G_j A_j^T & G_j + G_j^T - \sum_{i=1}^{N} f^j_i P_i & \eta I \\
    0 & 2 \eta I & \star & \star \\
\end{bmatrix} = \Theta_{j} > 0 \quad (32)
\]

for $j = 1, \ldots, M$ and $\ell = j+1, \ldots, M$, then the
system $H$ is exponentially stable and
\[
\|H\| \leq \min_{\ell_k, \gamma_j} \eta
\]

**Proof:** Take any $\gamma \in \Lambda_M$. Now, multiply (31) by $\gamma_j^2$ and sum for $j = 1, \ldots, M$. Likewise, multiply (32) by $\gamma_j$ and sum for $j = 1, \ldots, M - 1$ and $\ell = j + 1, \ldots, M$. Adding the resulting two expressions yields
\[
\hat{\Theta} = \sum_{j=1}^{M} \gamma_j^2 \Theta_j + \sum_{j=1}^{M-1} \sum_{\ell=j+1}^{M} \gamma_j \gamma_\ell \Theta_{j\ell}
\]
\[
= \begin{bmatrix}
\tilde{P}(\gamma) & \star & \star & \star \\
G(\gamma)^T A(\gamma)^T & G(\gamma) + G(\gamma)^T - \tilde{P}(\gamma) & \star & \star \\
\tilde{B}_w(\gamma)^T & 0 & \eta I & \star \\
0 & \tilde{C}(\gamma) G(\gamma) & \tilde{D}_w(\gamma) & \eta I
\end{bmatrix}
\]

which is, due to (24) and under the specific parameterisations (26)–(30), equivalent to $\Theta$ in (25). Feasibility of the LMIs (31) and (32) ensures that $\hat{\Theta} > 0$ for all $\gamma(\ell) \in \Lambda_M$ with $k \geq 0$. Consequently, the condition of Theorem 4 is satisfied for all $\alpha(\ell) \in \Lambda_N$ with $k \geq 0$. $\square

### 4.3 $H_2$ Performance

The LMI condition in Theorem 3 is particularised for polytopic LTV systems in a similar way as done for the $H_\infty$ performance.

**Theorem 6:** Consider the system $H$ given by (16). If there exist parameter-dependent matrices $G(\alpha(\ell)), P(\alpha(\ell)) = P(\alpha(\ell))^T > 0$ and $W(\alpha(\ell)) = W(\alpha(\ell))^T > 0$, for all $\alpha(\ell) \in \Lambda_N$, such that
\[
\begin{bmatrix}
P(\alpha(k + 1)) & \star & \star \\
G(\alpha(\ell))^T A(\alpha(\ell))^T & G(\alpha(\ell)) + G(\alpha(\ell))^T - P(\alpha(\ell)) & \star \\
\tilde{B}_w(\alpha(\ell))^T & 0 & I
\end{bmatrix}
= \Phi > 0
\]

(33)

\[
\begin{bmatrix}
W(\alpha(\ell)) - D_z(\alpha(\ell)) D_w(\alpha(\ell))^T & \star \\
G(\alpha(\ell))^T C_z(\alpha(\ell))^T & G(\alpha(\ell)) + G(\alpha(\ell))^T - P(\alpha(\ell))
\end{bmatrix}
= \Psi > 0
\]

(34)

then the system $H$ is exponentially stable and its $H_2$ performance is bounded by $v$ given by
\[
v^2 = \inf_{P(\alpha(\ell)), G(\alpha(\ell)), H(\alpha(\ell)), \alpha(\ell) \in \Lambda_N} \sup_{\alpha(\ell) \in \Lambda_N} \text{Tr}[W(\alpha(\ell))]
\]

(35)

The proof follows directly from LMI conditions (14) and (15) in Theorem 3 by considering the specific time dependency on the time-varying parameter $\alpha(\ell)$ of system (16) and by noticing that
\[
\frac{1}{T} \sum_{k=0}^{T} \text{Tr}[W(\alpha(\ell))] \leq \sup_{\alpha(\ell) \in \Lambda_N} \text{Tr}[W(\alpha(\ell))]
\]

which implies
\[
\inf_{P(\alpha(\ell)), G(\alpha(\ell)), H(\alpha(\ell)), \alpha(\ell) \in \Lambda_N} \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \text{Tr}[W(\alpha(\ell))]
\]

\[
\leq \inf_{P(\alpha(\ell)), G(\alpha(\ell)), H(\alpha(\ell)), \alpha(\ell) \in \Lambda_N} \sup_{\alpha(\ell) \in \Lambda_N} \text{Tr}[W(\alpha(\ell))]
\]

(36)

Again, the conditions of Theorem 6, which consist of evaluating the parameter-dependent LMIs for all $\alpha(\ell)$ in the unit simplex $\Lambda_N$, lead to an infinite dimensional problem. However, in the same way as for the $H_\infty$ performance, a finite-dimensional set of LMIs can be obtained. This is presented in the next theorem.

**Theorem 7:** Consider system $H$, given by (16). Assume that the vectors $f^j$ and $h^j$ of $\Gamma_j$ are given. If there exist, for $j = 1, \ldots, M$, matrices $G_j \in \mathbb{R}^{n \times n}$ and, for $i = 1, \ldots, N$, symmetric positive-definite matrices $P_i \in \mathbb{R}^{n \times n}$ and $W_i \in \mathbb{R}^{n \times p}$ such that
\[
\begin{bmatrix}
\sum_{i=1}^{N} (f_i^j + h_i^j) P_j & \star & \star \\
G_j^T A_j^T & G_j + G_j^T - \sum_{i=1}^{N} f_i^j P_i & \star \\
\tilde{B}_w^T & 0 & I
\end{bmatrix}
= \Phi_j > 0
\]

(37)

for $j = 1, \ldots, M$ (see (37))

\[
\begin{bmatrix}
\sum_{i=1}^{N} f_i^j W_i - \tilde{D}_w D_w^T & \star \\
G_j^T C_j^T & G_j + G_j^T - \sum_{i=1}^{N} f_i^j P_i
\end{bmatrix}
= \Psi_j > 0
\]

(38)

for $j = 1, \ldots, M - 1$ and $\ell = j + 1, \ldots, M$
for \( j = 1, \ldots, M \) (see (39))

for \( j = 1, \ldots, M - 1 \) and \( \ell = j + 1, \ldots, M \), then the system \( H \) is exponentially stable and its \( \mathcal{H}_2 \) performance is bounded by \( v \) given by

\[
v^2 = \min_{P, \gamma_1, \gamma_2} \max_i \text{Tr}[W_i]
\]

Proof: Take any \( \gamma \in \Lambda_M \). Now, multiply (36) by \( \gamma^2 \) and sum for \( j = 1, \ldots, M \). Likewise, multiply (37) by \( \gamma_j \) and sum for \( j = 1, \ldots, M - 1 \) and \( \ell = j + 1, \ldots, M \). Adding the resulting two expressions yields

\[
\hat{\Phi} = \sum_{j=1}^{M} \gamma_j^2 \Phi_j + \sum_{j=1}^{M-1} \sum_{\ell=j+1}^{M} \gamma_j \gamma_\ell \Phi_j \ell
\]

which is, due to (24) and under the specific parameterisations (26)–(30), equivalent to \( \Phi \) in (33). Analogously, multiply (38) by \( \gamma \) and sum for \( j = 1, \ldots, M \). Multiply (39) by \( \gamma_j \) and sum for \( j = 1, \ldots, M - 1 \) and \( \ell = j + 1, \ldots, M \). Adding both results yields

\[
\hat{\Psi} = \sum_{j=1}^{M} \gamma_j^2 \Psi_j + \sum_{j=1}^{M-1} \sum_{\ell=j+1}^{M} \gamma_j \gamma_\ell \Psi_j \ell
\]

which is, due to (24) and under the specific parameterisations (26)–(30), equivalent to \( \Psi \) in (34). Feasibility of the LMIs (36)–(39) ensures that \( \hat{\Phi} > 0 \) and \( \hat{\Psi} > 0 \) for all \( \gamma \in \Lambda_M \) with \( k \geq 0 \). Consequently, the conditions of Theorem 6 are satisfied for all \( \alpha(k) \in \Lambda_N \) with \( k \geq 0 \). Moreover, since \( \alpha \in \Lambda_N \), it follows that

\[
\sup_{\alpha \in \Lambda_N} \text{Tr}[W(\alpha)] = \max_i \text{Tr}[W_i] \tag{40}
\]

and therefore

\[
\inf_{P(\alpha), G(\alpha), W(\alpha)} \sup_{\alpha \in \Lambda_N} \text{Tr}[W(\alpha)] \leq \min_i \max \text{Tr}[W_i] \nonumber
\]

\[\square\]

### 5 Gain-scheduled \( \mathcal{H}_\infty \) static output feedback

In this section, the analysis results presented in Theorems 5 and 7 are extended to provide a finite set of LMI conditions for the synthesis of \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) guaranteed performance gain-scheduled static output feedback controllers for the system given by (16).

It is assumed that the first \( q \) states of the system can be measured in real-time for feedback without corruption by the exogenous input \( w(k) \) or the control input \( u(k) \), that is

\[
y(k) = C_2 x(k), \quad C_j = \begin{bmatrix} I_q & 0_{n-q,q} \end{bmatrix}
\] (41)

If this is not the case, one can use a similarity transformation as proposed in [33], whenever the output matrix \( C_j \) is not affected by the time-varying parameter.

The goal is to provide a parameter-dependent control law

\[
u(k) = K(\alpha(k))y(k), \quad \text{with} \quad K(\alpha(k)) \in \mathbb{R}^{m \times q} \tag{42}
\]

such that the closed-loop system

\[
\begin{cases}
x(k+1) = A_d(\alpha(k))x(k) + B_u(\alpha(k))w(k) \\
y(k) = C_j(\alpha(k))x(k) + D_u(\alpha(k))w(k)
\end{cases}
\]

is exponentially stable and has a guaranteed \( \mathcal{H}_\infty \) or \( \mathcal{H}_2 \) performance for all possible variation of the parameter \( \alpha(k) \in \Lambda_N \).

#### 5.1 Gain-scheduled \( \mathcal{H}_\infty \) static output feedback

A solution to the gain-scheduled \( \mathcal{H}_\infty \) static output feedback design problem, in terms of a finite set of LMIs, is provided by the next theorem.

**Theorem 8:** Consider system \( H \), given by (16). Assume that the vectors \( f^i \) and \( h^i \) of \( \Gamma_k \) are given. If there exist, for \( i = 1, \ldots, N \), matrices \( G_{i,1} \in \mathbb{R}^{p \times q}, Z_{i,1} \in \mathbb{R}^{m \times q} \) and symmetric positive-definite matrices \( P_i \in \mathbb{R}^{n \times n} \), and, for \( j = 1, \ldots, M \), matrices \( G_{j,2} \in \mathbb{R}^{(n-q) \times q}, G_{j,3} \in \mathbb{R}^{(n-q) \times (n-q)} \) such that

\[
\begin{bmatrix}
\sum_{i=1}^{N} (f^i + h^i) P_i & \ast & \ast \\
G^T A_d(\alpha(k)) & G & G^T - \sum_{i=1}^{N} f^i P_i & \ast & \ast \\
B_{u,d} & 0 & \eta I & \ast & \ast \\
0 & \tilde{C}_{x,d} & \tilde{D}_{u,d} & \eta I & \ast
\end{bmatrix} = \Theta_j > 0
\] (44)

\[
\begin{bmatrix}
\sum_{i=1}^{N} (f^i + f^i_2) W_i - \tilde{D}_{u,d} & \tilde{D}_{u,d} & \tilde{D}_{u,d}^T & \ast & \ast \\
G^T \hat{C}_{x,d} + G^T & G^T - \sum_{i=1}^{N} f^i P_i & \ast & \ast & \ast \\
0 & \tilde{C}_{x,d} & G_j & \tilde{D}_{u,d} & \eta I
\end{bmatrix} = \Psi_j > 0
\] (39)
for \( j = 1, \ldots, M \) and

\[
\begin{bmatrix}
\sum_{i=1}^{N} (f_i^j + f_i'^j + b_i^j + b_i'^j)P_i \\
\Theta_{21,j} \\
B_{w,j}^T + B_{w,j}'^T \\
0
\end{bmatrix} \quad \begin{bmatrix}
\Theta_{22,j} \\
0 \\
2\eta \ell
\end{bmatrix} = \begin{bmatrix}
\Theta_{42,j} \\
D_{w,j} + D_{w,j}'
\end{bmatrix} \eta I
\]

with

\[
\Theta_{21,j} = G_j^T A_i + G_i^T A_j + Z_i^T B_{w,i} + Z_j^T B_{w,j} \\
\Theta_{22,j} = G_j + G_i + G_j^T - \sum_{i=1}^{N} (f_i^j + f_i'^j)P_i \\
\Theta_{42,j} = \tilde{C}_{x,e}^j G_i + \tilde{C}_{x,e}^j G_j + \tilde{D}_{w,i} Z_i + \tilde{D}_{w,i}' Z_j,
\]

for \( j = 1, \ldots, M - 1 \) and \( \ell = j + 1, \ldots, M \), where

\[
G_j = \begin{bmatrix}
\sum_{i=1}^{N} f_i^j G_{i,1} \\
0 \\
G_{i,j}
\end{bmatrix} \quad \text{and} \quad Z_j = \begin{bmatrix}
\sum_{i=1}^{N} f_i'^j Z_{i,1} \\
0
\end{bmatrix}
\]

then the parameter-dependent static output feedback gain

\[
K(\alpha) = \tilde{Z}(\alpha)\hat{G}(\alpha)^{-1}
\]

with

\[
\tilde{Z}(\alpha) = \sum_{i=1}^{N} \alpha_i Z_{i,1} \quad \text{and} \quad \hat{G}(\alpha) = \sum_{i=1}^{N} \alpha_i G_{i,1}
\]

stabilises the system \( H \) with a guaranteed \( \mathcal{H}_\infty \) performance bounded by \( \eta \).

**Proof:** Take any \( \gamma \in \Lambda_M \). Now, multiply (44) by \( \gamma \) and sum for \( j = 1, \ldots, M \). Likewise, multiply (45) by \( \gamma \gamma_i \) and sum for \( j = 1, \ldots, M - 1 \) and \( \ell = j + 1, \ldots, M \). Adding the resulting two expressions yields (see equation at the bottom of the page)

with \( G(\gamma) = \sum_{i=1}^{M} \gamma_i G_i \) and \( Z(\gamma) = \sum_{i=1}^{M} \gamma_i Z_i \) with \( G_i \) and \( Z_i \) given by (46). Now, using (24), (41), (43), (47) and (48), it follows that

\[
\tilde{A}(\gamma) G(\gamma) + \tilde{B}_\gamma(\gamma) Z(\gamma) = \begin{bmatrix}
\tilde{A}(\alpha) G(\gamma) + B_\gamma(\alpha) \tilde{Z}(\alpha) & 0 \\
\tilde{A}(\alpha) & 0
\end{bmatrix} \hat{G}(\alpha) = \begin{bmatrix}
G(\gamma) + \tilde{G}(\gamma) & 0 \\
0 & G_2(\gamma)
\end{bmatrix}
\]

Therefore the above LMI expression can be written as

\[
\tilde{\Theta} = \begin{bmatrix}
\tilde{P}(\gamma) \\
G(\gamma)^T A_i(\alpha) & 0 \\
0 & C_\alpha(\gamma) D_\alpha(\gamma)
\end{bmatrix} \begin{bmatrix}
\tilde{P}(\gamma) \\
0 \\
\eta I
\end{bmatrix}
\]

which is, due to (24) and under the specific parameterisations (26)–(30), equivalent to \( \Theta \) in (25) for the closed-loop system (42). Feasibility of the LMIs (44) and (45) ensures that \( \tilde{\Theta} > 0 \) for all \( \gamma(\alpha) \in \Lambda_M \) with \( k \geq 0 \). Consequently, the condition of Theorem 4 is satisfied for the closed-loop system (42) for all \( \alpha(\alpha) \in \Lambda_M \) with \( k \geq 0 \).

### 5.2 Gain-scheduled \( \mathcal{H}_2 \) static output feedback

Consider the discrete-time LTV system given by (16). A solution to the gain-scheduled \( \mathcal{H}_2 \) static output feedback design problem, in terms of a finite set of LMIs, is provided by the next theorem.

**Theorem 9:** Consider system \( H \), given by (16). Assume that the vectors \( f^j \) and \( \hat{f} \) of \( \Gamma_k \) are given. If there exist, for \( i = 1, \ldots, N, \) matrices \( G_{i,1} \in \mathbb{R}^{p \times q} \), \( Z_i \in \mathbb{R}^{p \times q} \) and symmetric positive-definite matrices \( P_i \in \mathbb{R}^{n \times n} \) and \( W_i \in \mathbb{R}^{n \times p} \), and, for \( j = 1, \ldots, M \), matrices \( G_{j,2} \in \mathbb{R}^{(n-p) \times q} \) and \( K_j \in \mathbb{R}^{(n-q) \times q} \), then

\[
\tilde{\Theta} = \sum_{j=1}^{M} \gamma_j^2 \Theta_{j,1} + \sum_{j=1}^{M} \sum_{\ell=j+1}^{M} \gamma_j \gamma_\ell \Theta_{j,\ell}
\]

\[
= \begin{bmatrix}
\tilde{P}(\gamma) \\
G(\gamma)^T A_i(\alpha) & 0 \\
0 & C_\alpha(\gamma)
\end{bmatrix} \begin{bmatrix}
\tilde{P}(\gamma) \\
0 \\
\eta I
\end{bmatrix}
\]

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and $G_{j,3} \in \mathbb{R}^{(s-q) \times (s-q)}$ such that
\[
\begin{bmatrix}
\sum_{i=1}^{N} f_i^T + h_i^T \Phi_i \\
G_j^T A_j + Z_j^T \Phi_{aj} & G_j + G_j^T - \sum_{i=1}^{N} f_i^T \Phi_i \\
\bar{B}_{aj} & I
\end{bmatrix} > 0
\]
(49)
for $j = 1, \ldots, M$ (see (50))

\[
\begin{bmatrix}
\sum_{i=1}^{N} f_i^T W_i - \bar{D}_{ai} \bar{D}_{ai}^T \\
G_j^T C_{aj} + Z_j^T \bar{D}_{ai} & G_j + G_j^T - \sum_{i=1}^{N} f_i^T \Phi_i \\
\bar{B}_{ai} & 0
\end{bmatrix} > 0
\]
(51)
for $j = 1, \ldots, M$ (see (52))

\[
\begin{bmatrix}
\sum_{i=1}^{N} (f_i^T + f_i') W_i - \bar{D}_{ai} \bar{D}_{ai}^T \\
G_j^T A_j + Z_j^T \Phi_{aj} & G_j + G_j^T - \sum_{i=1}^{N} (f_i^T + f_i') \Phi_i \\
\bar{B}_{aj} & 0
\end{bmatrix} > 0
\]
(50)

and applying the control synthesis procedures of Theorem 8 or 9.

For time-invariant systems, conditions for the design of multiobjective controllers are presented in [34]. By fixing the Lyapunov matrix for each performance specification $j$, the problem is shown to be convex, at the cost of a conservative controller. As the same Lyapunov variable appears in all performance constraints, all freedom that is available in the Lyapunov variable is exploited by the different constraints and therefore this method is named the Lyapunov shaping paradigm. In [25], the introduction of slack variables in the control synthesis conditions is exploited to reduce the conservatism of the Lyapunov shaping paradigm, since this introduction implies that for every different performance specification, a different Lyapunov function can be used. Since in this case, all freedom that is available in the common slack variable $G$ is exploited by the different performance specifications, the method of [25] is called the $G$ shaping paradigm. As the slack variables are fixed for all performance specifications, the $G$ shaping paradigm still has some conservatism. However, the fact that a different set of Lyapunov variables can be used for each performance specification provides important control design freedom. Since the conditions of Theorems 8 and 9 can be combined using a common slack variable, it is clear that the $G$ shaping paradigm of [25] can be extended to the class of discrete-time polytopic LTV systems with bounds on the rate of parameter variation.

5.4 Robust static output feedback

Synthesis conditions for a robust static output feedback controller $u(k) = K \hat{y}(k)$ can be derived from Theorems 8 and 9 by enforcing the slack variables $\hat{G}(\alpha)$ and $\hat{Z}(\alpha)$ in the open-loop system (16), using for each performance specification $j$, for $j = 1, \ldots, s$ (where $s$ is the number of performance specifications), selection matrices $L_j$ and $M_j$ as follows

\[
H_j := \begin{cases}
x(k+1) = A(\alpha(k))x(k) + B_w(\alpha(k))M_j u(k) \\
+ B_w(\alpha(k))u(k) \\
z(k) = L_j C_j(\alpha(k))x(k) + L_j D_w(\alpha(k))M_j u(k) \\
+ L_j D_w(\alpha(k))u(k)
\end{cases}
\]
(54)

The proof of Theorem 9 is omitted since it can be constructed analogously to the proof of Theorem 8.

Note that if all states are available for feedback, that is, $y(k) = x(k)$, the LMIIs in Theorems 8 and 9 provide conditions for the existence of a gain-scheduled static state feedback control law

\[
u(k) = K(\alpha(k))x(k)
\]

5.3 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ gain-scheduled static output feedback

This section deals with the problem of finding a gain-scheduled static output feedback controller for system (16) such that multiobjective $\mathcal{H}_2$ and $\mathcal{H}_\infty$ closed-loop specifications are met. These specifications can be imposed on different closed-loop input–output combinations by appropriately selecting the right input–output channels of the open-loop system (16), using for each performance specification $j$, for $j = 1, \ldots, s$ (where $s$ is the number of performance specifications), selection matrices $L_j$ and $M_j$ as follows

\[
\begin{bmatrix}
\sum_{i=1}^{N} (f_i^T + f_i') W_i - \bar{D}_{ai} \bar{D}_{ai}^T \\
G_j^T A_j + Z_j^T \Phi_{aj} & G_j + G_j^T - \sum_{i=1}^{N} (f_i^T + f_i') \Phi_i \\
\bar{B}_{ai} & 0
\end{bmatrix} > 0
\]
(50)

\[
\begin{bmatrix}
\sum_{i=1}^{N} (f_i^T + f_i') W_i - \bar{D}_{aj} \bar{D}_{aj}^T \\
G_j^T C_{aj} + Z_j^T \bar{D}_{ai} & G_j + G_j^T - \sum_{i=1}^{N} (f_i^T + f_i') \Phi_i \\
\bar{B}_{aj} & 0
\end{bmatrix} > 0
\]
(52)
(48) to be parameter-independent, that is \( \hat{G}(\alpha) = \hat{G} \) and \( \hat{Z}(\alpha) = \hat{Z} \). Since \( \sum_{i=1}^{N} \alpha_{i} = 1 \), \( G(\alpha) \) and \( \hat{Z}(\alpha) \) can be parameterised as follows

\[
\hat{G}(\alpha) = \hat{G} = \left( \sum_{i=1}^{N} \alpha_{i} \right) \hat{G} = \sum_{i=1}^{N} \alpha_{i} \hat{G}
\]

\[
\hat{Z}(\alpha) = \hat{Z} = \left( \sum_{i=1}^{N} \alpha_{i} \right) \hat{Z} = \sum_{i=1}^{N} \alpha_{i} \hat{Z}
\]

This is the context of the following corollaries.

**Corollary 1:** Consider system \( H_{i} \) given by (16). Assume that the vectors \( f_{j} \) and \( h_{j} \) of \( \Gamma_{j} \) are given. If there exist matrices \( \hat{G} \in \mathbb{R}^{p \times q} \) and \( \hat{Z} \in \mathbb{R}^{m \times q} \), matrices \( G_{j} \in \mathbb{R}^{(n-q)\times q} \), \( G_{j,2} \in \mathbb{R}^{(n-q)\times(n-q)} \), for \( j = 1, \ldots, M \), and symmetric positive-definite matrices \( P_{i} \in \mathbb{R}^{n \times n} \), for \( i = 1, \ldots, N \), such that (44) holds for \( j = 1, \ldots, M \) and (45) holds for \( j = 1, \ldots, M - 1 \) and \( \ell = j + 1, \ldots, M \), with

\[
G_{j} = \begin{bmatrix} \hat{G} \\ G_{j,2} \\ G_{j,3} \end{bmatrix} \quad \text{and} \quad Z_{j} = \begin{bmatrix} \hat{Z} \\ 0 \end{bmatrix}
\]

then the robust static output feedback gain \( K = \hat{Z} \hat{G}^{-1} \) stabilises the system \( H \) with a guaranteed \( H_{\infty} \) performance bounded by \( \eta \).

**Corollary 2:** Consider system \( H_{i} \), given by (16). Assume that the vectors \( f_{j} \) and \( h_{j} \) of \( \Gamma_{j} \) are given. If there exist matrices \( \hat{G} \in \mathbb{R}^{p \times q} \) and \( \hat{Z} \in \mathbb{R}^{m \times q} \), matrices \( G_{j,2} \in \mathbb{R}^{(n-q)\times(n-q)} \), \( G_{j,3} \in \mathbb{R}^{(n-q)\times(n-q)} \), for \( j = 1, \ldots, M \), and symmetric positive-definite matrices \( P_{i} \in \mathbb{R}^{n \times n} \) and \( W_{i} \in \mathbb{R}^{p \times q} \), for \( i = 1, \ldots, N \), such that (49) holds for \( j = 1, \ldots, M \), (50) holds for \( j = 1, \ldots, M - 1 \) and \( \ell = j + 1, \ldots, M \), (51) holds for \( j = 1, \ldots, M \) and (52) holds for \( j = 1, \ldots, M - 1 \) and \( \ell = j + 1, \ldots, M \), with a prescribed bound \( \eta \) on the \( H_{\infty} \) performance from the exogenous disturbance \( w(k) \) to the first channel of the performance output \( z(k) \) under a prescribed bound \( \eta \) on the \( H_{\infty} \) performance from \( w(k) \) to the control signal \( u(k) \), defined as the second channel of \( z(k) \), as can be seen from the second row in \( C_{i,2} \), \( D_{i,2} \) and \( D_{i,4}, i = 1, \ldots, 4 \). The measured output is the first state of the system, that is, \( C_{i} = [1 \ 0 \ 0] \). This mixed \( H_{2}/H_{\infty} \) design is performed by combining the LMI conditions of Theorems 8 and 9, as explained in Section 5.3. To verify

\[
\begin{bmatrix} A_{1} & A_{2} & A_{3} & A_{4} \end{bmatrix} = 0.435 \cdot \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 2 & -2 \\ 2 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & -2 & -1 & 2 & 1 & 1 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} B_{u,1} & B_{u,2} & B_{u,3} & B_{u,4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} B_{w,1} & B_{w,2} & B_{w,3} & B_{w,4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}
\]

\[
\begin{bmatrix} C_{z,1} & C_{z,2} & C_{z,3} & C_{z,4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} D_{a,1} & D_{a,2} & D_{a,3} & D_{a,4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} D_{w,1} & D_{w,2} & D_{w,3} & D_{w,4} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]
the influence of the bound $b$ on the rate of variation, two cases are considered: $b = 0.3$ and $b = 0.4$. The vertices of the compact set (22) are generated using the approach shown in Appendix 11.3. For a system with $N = 4$ vertices, there are $M = 28$ vertices of the compact set (22).

All LMI optimisation problems are modelled by YALMIP [35] and solved using SeDuMi [36]. Fig. 2 shows the trade-off between the prescribed bound $\eta$ on the $H_\infty$ performance from the disturbance to the control signal and the obtained bound $\nu$ on the $H_2$ performance from the disturbance to the first channel of the performance output for the two considered bound cases: $b = 0.3$ (dark grey solid) and $b = 0.4$ (light grey dashed). For both cases it is clear that tighter control bounds (smaller values of $\eta$) result in a decrease in performance (higher values of $\nu$). Moreover, for small values of $\eta$ (indicated with a square), the set of LMIs becomes infeasible. The influence of the bound on the rate of variation is obvious: if the parameters are restricted to vary slower, the obtained upper bound $\nu$ on the performance decreases and the set of LMIs remains feasible for smaller values of $\eta$.

7 Vibroacoustic application

The goal of this application is to attenuate the structural noise of a vibroacoustic set-up whose dynamics is highly sensitive to the ambient temperature. Since the temperature variation is slow, taking physical bounds on this variation into account during the control design can reduce the conservatism typically associated with control synthesis procedures based on the quadratic stability concept that allows arbitrary fast parameter variation.

The set-up (displayed in Fig. 3) consists of a lexan plate clamped on a rigid baffle. For details see [37]. The disturbance $\omega$ is provided by a shaker, the control input $u$ by a piezoelectric patch and the output $z$ is the sound pressure measured by a microphone. For four different operating conditions $\tau \in \{22.9^\circ, 23.4^\circ, 24.4^\circ, 25.4^\circ\}$ (degrees Celsius), frequency response functions (FRFs) are measured from the inputs $w$ and $u$ to the output $z$ and then used to estimate four 10th-order discrete-time state-space models (Fig. 4). Based on these LTI models, a multiple input multiple output LPV model, that is affine in the temperature $\tau$, is derived using the State-space Model Interpolation of Local Estimates (SMILE) technique presented in [38, 39] (Fig. 5). This affine LPV model can be converted into a polytopic system (16) with two vertices ($N = 2$) by defining

$$\alpha_1 = \frac{\tau - \tau_{\min}}{\tau_{\max} - \tau_{\min}}, \quad \alpha_2 = 1 - \alpha_1, \quad \text{and} \quad 0 \leq \alpha_1 \leq 1$$

(56)

with $\tau_{\min} = 22.9^\circ$ and $\tau_{\max} = 25.4^\circ$.

7.1 Mixed $H_2/H_\infty$ control design

The aim is to minimise an upper bound $\nu$ on the closed-loop $H_2$ performance from the disturbance $\omega$ to the output $z$. To obtain realistic controllers, that do not have excessively large control signals, an upper bound $\eta$ is enforced on the $H_\infty$ cost.
from the disturbance $w$ to the control signal $u$. This mixed $H_2/H_\infty$ design is performed by combining the LMI conditions of Theorems 8 and 9, as explained in Section 5.3.

Both gain-scheduled and robust controllers are designed using the following settings:

- Measurement case: either 6 or 10 states are assumed available for feedback.
- Bound case: five bounds on the rate of variation $b \in \{0, 0.2, 0.5, 0.8, 1\}$.
- The upper bound $\eta$ on the closed-loop $H_\infty$ cost from the disturbance $w$ to the control signal $u$ is given by the range $\eta \in [0.8, 50]$.

Fig. 6 compares the results for measurement cases 6 and 10, by showing the trade-off between the prescribed $H_\infty$ bound $\eta$ and the obtained $H_2$ bound $\nu$. Solid lines indicate robust control designs (indicated R in the legend), whereas dashed lines indicate gain-scheduled control designs (indicated GS in the legend). For each measurement and bound case, an $H_2$ controller without considering any bound on the $H_\infty$ cost from the disturbance to the control signal was also calculated using the LMIs of Theorem 9. The obtained $H_2$ performance provides for each case the maximum achievable performance, that is, the minimal achievable value for $\nu$, indicated with $\nu$. Table 1 shows this maximum achievable performance for each measurement and bound case. Cases marked with an X indicate infeasible control design. In Fig. 6, the performance $\nu$ is indicated with a dash-dotted line for robust control designs and with a dotted line for gain-scheduled control designs.

From Fig. 6, several conclusions can be drawn. First, the mixed $H_2/H_\infty$ control designs always provide worse

### Table 1
Maximum achievable $H_2$ performance bound $\nu$

<table>
<thead>
<tr>
<th>Bound $b$</th>
<th>Number of states available for feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td>R</td>
<td>GS</td>
</tr>
<tr>
<td>GS</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.5430</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8290</td>
</tr>
<tr>
<td>0.5</td>
<td>X</td>
</tr>
<tr>
<td>0.8</td>
<td>X</td>
</tr>
<tr>
<td>1</td>
<td>X</td>
</tr>
</tbody>
</table>

$\nu$ | 0.1296 | 0.1233 |
| 0.3772 | 0.3395 |
| 0.4601 | 0.4210 |
| 0.5006 | 0.4704 |
| 0.5233 | 0.4981 |

**Fig. 5** Bode magnitude plot of the LPV model from $u$ to $z$, evaluated at 11 different temperatures (black), and the four estimated models (grey)

**Fig. 6** Trade-off between the prescribed bound $\eta$ and the obtained performance $\nu$

- $a$ Six states measured
- $b$ Ten states measured
performance bounds $v$ compared to the $H_2$ control design. Second, as expected, the gain-scheduled controllers always perform better than their robust counterparts: on the plots the dashed lines are always below the solid lines. Of course, this also holds for the maximum achievable performance $v$ as can be seen by comparing the columns of Table 1 for every measurement case. Third, it is clear that as the bound $b$ on the rate of variation increases, the performance decreases, which is also intuitive. Moreover, for measurement case 6, the LMI conditions for the bound cases which is also intuitive. Moreover, for measurement case 6, the maximum bound $h_{max} = 0.224761$ is found for both the robust and gain-scheduled control design. The fourth conclusion is that as more states become available for feedback, the performance gets better. This can be seen by comparing lines of the same colour between Figs. 6a and b.

7.2 Comparison to existing results in literature

In [20] a synthesis procedure for gain-scheduled $H_\infty$ static state feedback controllers, based on a piecewise constant Lyapunov function, is presented for discrete-time multi-affine LPV systems with bounded parameter variation. A bound on the rate of variation is imposed as follows (here, only the basic procedure for the single parameter case is presented). First, the parameter $p$ is assumed to belong to a finite interval $[\bar{p}, \tilde{p}]$. Then this parameter space is divided in $a$ intervals of equal length $((\tilde{p} - \bar{p})/a)$ and, in one time step, the parameter $p$ is only allowed to move to a neighbouring interval. Based on these assumptions and restrictions, a finite set of LMI is derived for the synthesis of a gain-scheduled $H_\infty$ static state feedback controller.

To apply the synthesis procedures of [20, Theorem 6], the polytopic LPV model is transformed again to an affine LPV model in $\alpha$, defined in (56). Since the polytopic LPV model only has two vertices, this transformation is exact. As $0 \leq \alpha \leq 1$, the number $a$ of intervals corresponds to a bound $b = 1/a$ on the rate of parameter variation. Since the results in [20] are restricted to $H_\infty$ performance, the closed-loop $H_\infty$ performance, denoted by $\gamma$, is minimised, while considering an $H_\infty$ bound $\eta$ on the control effort.

The results are shown in Fig. 7, which shows the trade-off between the prescribed $H_\infty$ bound on the control signal $\eta$ and the obtained $H_\infty$ performance $\gamma$. It is clear that the gain-scheduled controllers obtained with the LMI conditions from Theorem 8 (dashed lines) show better performance than the controllers obtained using [20] (solid lines). Indeed, to obtain the same guaranteed performance $\gamma$, the bound on the control signal $\eta$ is considerably smaller for the controllers obtained using Theorem 8 and when imposing the same bound $\eta$, a better closed-loop performance, that is, a smaller value for $\gamma$, is guaranteed.

Table 2 Comparison of the numerical complexity

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>Solver time (s)</th>
<th>$V$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>8.67</td>
<td>1.82</td>
<td>441</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>27.3</td>
<td>4.48</td>
<td>441</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>23.7</td>
<td>29.6</td>
<td>441</td>
</tr>
<tr>
<td>20</td>
<td>0.05</td>
<td>26.4</td>
<td>67.9</td>
<td>441</td>
</tr>
</tbody>
</table>
where \( n \) is the number of states, \( m \) the number of control inputs, \( r \) the number of disturbances, \( i \) the number of performance specifications, \( N \) the number of vertices for the polytopic system, \( M \) the number of vertices of the uncertainty domain (22) and \( a \) is the number of intervals in the synthesis procedure from [20]. Note that the formulas for \( V_{\text{Amato}} \) and \( R_{\text{Amato}} \) only hold for affine dependency on a single parameter which is restricted to jump only one interval each time step (see [20] for more details). The values for the parameters in (57) are \( n = 10, m = 1, r = 1, s = 2 \) (performance specifications: \( z_1 \) is the control effort, \( z_2 \) the microphone output), \( N = 2, M = 6 \) and \( a \in \{1, 2, 10, 20\} \).

Table 2 clearly shows that for the synthesis procedure of Theorem 8, the number of variables \( V \) and the number of LMI rows \( R \) are independent of the bound \( b \) on the rate of parameter variation, while for the synthesis procedure of Theorem 8 in [20], \( V \) and \( R \) increase very fast as the number of intervals \( a \) increases. This, of course, has a clear reflection in the computation time of the solver, as can be seen by comparing the first two columns.

For this example, it can be concluded that, compared to [20], less conservative results can be obtained with less computational burden. This clearly shows the potential of the proposed synthesis procedures.

8 Conclusion

In this work, new LMI conditions are presented for the synthesis of robust and gain-scheduled \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) static output feedback controllers for discrete-time polytopic LTV systems based on parameter-dependent Lyapunov functions. The synthesis procedures explicitly take an a priori known bound on the rate of parameter variation into account, thus reducing the conservatism generally associated with methods that allow arbitrarily fast parameter variation. Moreover, an extension of the \( G \) shaping paradigm, presented for time-invariant systems in [25], is proposed here for the time-varying case, which allows the design of mixed \( \mathcal{H}_2/\mathcal{H}_\infty \) controllers using a different Lyapunov function for each performance specification. It is worth to emphasise that when the parameter is assumed uncertain, but frozen, Corollaries 1 and 2 reduce to the robust full state feedback case of [25, Theorems 10 and 9].

The new synthesis procedures have been used to design robust and gain-scheduled static output feedback controllers for a realistic vibroacoustic set-up, whose dynamics change considerably when the ambient temperature varies. As the rate of temperature variation is obviously bounded, the application of synthesis procedures that consider bounds on rate of variation is particularly interesting in this control problem. For the presented application, compared to [20], better performance is achieved using less computational resources.

9 Acknowledgments

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10 References


11 Appendix

11.1 Proof of Lemma 1

This section proves the characterisation of the infinite horizon $\mathcal{H}_2$ performance, as presented in Lemma 1. First, the finite horizon $\mathcal{H}_2$ performance is defined and a characterisation is provided to compute it. Later, this result is extended to the infinite horizon case.

Definition 2: Finite horizon $\mathcal{H}_2$ performance of a discrete-time LTV system.

Consider the system $H$, given by (1). Its $\mathcal{H}_2$ performance over a finite horizon $k \in [0, T]$ is defined as

$$\|H\|_{2,[0,T]}^2 = \delta \left\{ \frac{1}{T} \sum_{k=0}^{T} z(k)^T z(k) \right\}$$

(58)

when $z(k)$ is a zero-mean white noise Gaussian process with identity covariance matrix, that is

$$\delta [w(j)w(l)^T] = \delta_{jl} I,$$

The next lemma shows how the finite horizon $\mathcal{H}_2$ performance, defined in (58) can be computed.

Lemma 2: Consider the system $H$, given by (1). Its $\mathcal{H}_2$ performance over a finite horizon $k \in [0, T]$ is characterised as

$$\|H\|_{2,[0,T]}^2 = \frac{1}{T} \sum_{k=0}^{T} \text{Tr}(C_k(k)\tilde{P}(k)C_k(k)^T + D_w(k)D_w(k)^T)$$

(59)

$$= \frac{1}{T} \sum_{k=0}^{T} \text{Tr}(B_w(k)^T \tilde{Q}(k+1)B_w(k) + D_w(k)^T D_w(k))$$

(60)

in which $\tilde{P}(k)$ is the controllability Gramian satisfying

$$\tilde{P}(k+1) = A(k)\tilde{P}(k)A(k)^T + B_w(k)B_w(k)^T, \quad \tilde{P}(0) = 0$$

(61)

and $\tilde{Q}(k)$ is the observability Gramian satisfying

$$\tilde{Q}(k) = A(k)^T \tilde{Q}(k+1)A(k) + C_w(k)^T C_w(k), \quad \tilde{Q}(T+1) = 0$$

(62)

Proof: The output $z(k)$ of system (1) can be calculated as

$$z(k) = C_w(k) \left[ \sum_{j=0}^{k-1} \left( \sum_{i=j+1}^{k-1} A(i) \right) B_w(j) w(j) \right] + B_w(k-1) w(k-1) + D_w(k) w(k)$$

which can be written as

$$z(k) = \sum_{j=0}^{k} G(k,j) w(j)$$

(63)

with $G(k,j)$ given by (see (64))

$$G(k,j) = \begin{cases} C_w(k) \left( \prod_{i=j+1}^{k-1} A(i) \right) B_w(j), & \text{if } j \neq k \\ D_w(k), & \text{if } j = k \end{cases}$$

(64)

Since $z(k)^T z(k) = \text{Tr}(z(k)z(k)^T)$, substituting (63) in (58) yields

$$\|H\|_{2,[0,T]}^2 = \delta \left\{ \frac{1}{T} \sum_{k=0}^{T} \text{Tr} \left( \sum_{j=0}^{k} G(k,j) w(j)^T \left( \sum_{l=0}^{k} w(l)^T G(k,l)^T \right) \right) \right\}$$

$$= \delta \left\{ \frac{1}{T} \sum_{k=0}^{T} \text{Tr} \left( \sum_{j=0}^{k} G(k,j) w(j)^T w(l)^T G(k,l) \right) \right\}$$

$$= \frac{1}{T} \sum_{k=0}^{T} \text{Tr} \left( \sum_{j=0}^{k} \sum_{l=0}^{k} G(k,j) \delta [w(j)w(l)^T] G(k,l) \right)$$

(64)

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Since \( w(k) \) is a zero-mean white noise Gaussian process with covariance matrix \( \delta \{ w(j)w(j)^T \} = \delta_{ij}I \),
\[
\|H\|_{L_2[0,T]}^2 = \frac{1}{T} \sum_{k=0}^{T-1} \text{Tr} \left\{ \sum_{j=0}^{k-1} G(k,j)G(k,j)^T \right\}
\]  
(65)

Substituting (64) in (65) leads to
\[
\|H\|_{L_2[0,T]}^2 = \frac{1}{T} \sum_{k=0}^{T-1} \text{Tr} \left\{ \sum_{j=0}^{k-1} C_j(k) \left( \prod_{i=j}^{k-1} A(i) \right) B_w(j)B_w(j)^T \times \left( \prod_{i=j+1}^{k} A(i)^T \right) C_z(k)^T + D_{w}(k)D_{w}(k)^T \right\}
\]

Consequently
\[
\sum_{k=0}^{T} \text{Tr} \left( C_z(k)\bar{P}(k)C_z(k)^T + D_{w}(k)D_{w}(k)^T \right)
\]
\[
= \sum_{k=0}^{T} \text{Tr} \left( B_w(k)^T \bar{Q}(k+1)B_w(k) + D_{w}(k)^T D_{w}(k) \right)
\]
which proves (60) holds.

By extending the definition of the finite horizon \( \mathcal{H}_2 \) performance to the infinite horizon case, as presented in Definition 1, the proof of Lemma 1 follows straightforward from Lemma 2 and its proof.

### 11.2 Proof of Theorem 2

This section proves the LMI conditions of Theorem 2 that characterise an upper bound on the infinite horizon \( \mathcal{H}_2 \) performance of a discrete-time LTV system.

**Proof:** It follows from Lemma 1 that there exist bounded matrices \( \bar{P}(k) \) such that
\[
\bar{P}(k+1) = A(k)\bar{P}(k)A(k)^T + B_w(k)B_w(k)^T, \quad \text{with} \quad \bar{P}(0) = 0
\]
Since (8) implies that
\[
P(k+1) > A(k)P(k)A(k)^T + B_w(k)B_w(k)^T
\]
there exist matrices \( M(k) = M(k)^T > 0 \) such that
\[
P(k+1) = A(k)P(k)A(k)^T + B_w(k)B_w(k)^T + M(k)
\]
Consequently, \( P(k) > \bar{P}(k) \) for all \( k \geq 0 \). From (9) it follows that
\[
W(k) > C_z(k)\bar{P}(k)C_z(k)^T + D_{w}(k)D_{w}(k)^T
\]
\[
> C_z(k)\bar{P}(k)C_z(k)^T + D_{w}(k)D_{w}(k)^T
\]
Therefore
\[
\inf_{P(0),W(0)} \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \text{Tr} \{ W(k) \}
\]
\[
\geq \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \text{Tr} \{ C_z(k)\bar{P}(k)C_z(k)^T + D_{w}(k)D_{w}(k)^T \}
\]
\[
+ \|H\|_{L_2[0,T]}^2
\]
The proof of part (b) is analogous.
11.3 Generating the vertices of $\Gamma_b$

For a polytopic system (16) with system matrices varying in the polytope (17) with $N$ vertices and a given value of the bound $\delta$, the vertices of the uncertainty set $\Gamma_b$ can be generated as the columns of a matrix $V$ as follows:

$$V = \text{zeros}(2\times N, N\times (N-1));$$

FOR $i = 1:1:N$,

$$V(i,(i-1)\times N+1) = 1;$$

$\text{ind} = 1;$

FOR $j = 1:1:N$,

IF $j \neq i$

$$V([i \ N+i \ N+j],(i-1)\times N+\text{ind}+1) = [1 -b \ b ]';$$

$$V([i \ j \ N+i \ N+j],N^2+(i-1)\times (N-1)+\text{ind}) = [b \ 1-b\ -b \ b ]';$$

$\text{ind} = \text{ind}+1;$

END

END

END